On the derivation of effective gradient systems via EDP-convergence

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Abstract

In the realm of generalized gradient systems and metric gradient systems we study a notion of convergence suited for gradient flows which depend on a small parameter. This notion is called EDP-convergence. In order to understand the convergence of gradient systems we need an algorithm to derive the limiting energy as well as the limiting dissipation potential. By virtue of theory of Γ-convergence it is well understood how to compute the limit energy. However, it is the fundamental question of evolutionary Γ-convergence how to compute the limit dissipation potential.

The aim of this thesis is to show that EDP-convergence connects the microscopic dissipation potential with the macroscopic, i.e. limiting, dissipation potential in a meaningful and unique way. As a proof of concept 3 different examples are presented: (i) the diffusion equation on a thin sandwich-like domain, (ii) the porous medium equation with a thin interface and (iii) a wiggly energy model.

We recall already existing variational tools. In particular, the concept of gradient flows and their formulations in a Banach space setting as well as a metric setting. We show how the gradient flow concept that is used in this thesis can be used to obtain also gradient flows with respect to the Wasserstein metric. We motivate the definition of relaxed EDP-convergence and EDP-convergence with tilting. EDP-convergence is based upon the principle that there is an energy-dissipation-balance involving the total dissipation functional and the energy difference – the energy-dissipation-principle (EDP). The limit passage, in both the energy and the total dissipation functional, is performed in terms of Γ-convergence. In particular, general curves are considered, not only the solution to the gradient flow. By perturbing the flow as well as the driving force, the dissipation-landscape is explored and a kinetic relation for the limit system can be established.

The wiggly energy model demonstrates the importance of the kinetic relation for the construction of the limiting dissipation potential and thus the introduction of tilts. The models with a Wasserstein dissipation show that the limiting dissipation potential is not the naive limit. In particular, classical gradient systems with a quadratic dissipation potential converge to a generalized gradient systems. Methods are applied and developed in a standard Wasserstein-space setting, a Wasserstein-space setting with nonlinear mobility and a Hilbert-space setting.

Zusammenfassung

Diese Dissertation beschäftigt sich mit EDP-Konvergenz. Dabei handelt es sich um einen Konvergenzbegriff auf dem Gebiet der verallgemeinerten Gradientensysteme und metrischen Gradientensysteme, der geeignet ist für Gradientenflüsse, die von einem kleinen Parameter abhängen. EDP-Konvergenz liefert einen Algorithmus, der es erlaubt in der Energie und dem Dissipationspotenzial zum Grenzwert überzugehen. Durch die Γ-Konvergenz-Theorie ist verstanden, wie die Limes-Energie zu berechnen ist. Es ist die fundamentale Frage evolutionärer Γ-Konvergenz, wie das Limes-Dissipationspotenzial berechnet werden kann.

Das Ziel dieser Arbeit ist es aufzuzeigen, dass EDP-Konvergenz das mikro- und das makroskopische Dissipationspotenzial in einer sinnvollen und eindeutigen Art und Weise in Beziehung setzt. Anhand von drei Beispielen wird der Konvergenzbegriff untersucht: die Diffusionsgleichung auf einem dünnen, dreischichtigen Gebiet, die Poröse-Medien-Gleichung mit einer dünnen Membran und ein Modell mit oszillierender Energie.

Sowohl in Banach- als auch metrischen Räumen werden Gradientenflüsse und ihre Formulierungen eingeführt. Wir zeigen, dass die Formulierung mittels Energien und Dissipationspotenzialen auch die metrischen Wasserstein-Formulierungen erzeugt. Es wird die Definition von relaxierter EDP-Konvergenz und EDP-Konvergenz mit Kippung motiviert. EDP-Konvergenz basiert auf dem Prinzip, dass es ein Gleichgewicht zwischen Energie und Dissipation gibt – das Energie-Dissipations-Prinzip (EDP). Mittels Γ-Konvergenz wird sowohl in der Energie, als auch dem totalen Dissipationsfunktional zum Grenzwert übergegangen. Insbesondere spielen nicht nur die Lösungen der Gradientenflüsse ein Rolle, sondern beliebige Kurven. Durch die zusätzliche Entkopplung von Zustand und Triebkraft wird die Dissipationslandschaft erkundet und die kinetische Beziehung des Limessystems ermittelt.

Das Modell mit oszillierender Energie zeigt die Bedeutung der kinetischen Beziehung – und damit der Kippung – für die Herleitung des Limes-Dissipationspotenzials auf. Die Modelle mit Wasserstein-Dissipation zeigen, dass das Limes-Dissipationspotenzial nicht der naive Grenzwert ist. Insbesondere können klassische Gradientensysteme mit quadratischer Dissipation zu verallgemeinerten Gradientensysteme konvergieren. In dieser Arbeit werden Methoden für Wasserstein-Flüsse mit linearer und nicht-linearer Mobilität und für Hilbert-Räume genutzt und entwickelt.

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Contents

1 Introduction

When considering evolution in a micro-structure it is convenient for the sake of simplification to derive a coarse grained evolution equation. While there is a huge area of applications (e.g. [CCN+19, ZW17, ILU+13, PK09, Vot08, NCA+08]) we need to restrict ourselves to a special structure generating the microscopic evolution equation. With restriction to gradient flows we are able to rigorously justify the limit passage from the microscopic model to the macroscopic model. Moreover, by applying the method *EDP-convergence with tilting* we do not only pass to the limit in the evolution equation but additionally to the limit in the special structure, i.e. the gradient system inducing the gradient flow. The main contribution of this thesis is to give an algorithm for the derivation of the effective gradient system. The algorithm is demonstrated for five gradient systems. One is of wiggly energy type, one is a purely quadratic Hilbert-structure and the remaining three have a Wasserstein-structure.

Recent research in the field of gradient flows was initiated by [JKO98]. There is an extensive solution theory for gradient flows both in Banach spaces [MRS13, CV90, Col92] and metric spaces [RMS08, AGS05] where the latter elaborates on the Wasserstein-setting in particular. The Sandier-Serfaty approach [SS04] succeeds under restrictive assumptions to pass to the limit in the De Giorgi's *energy dissipation balance* (EDB) formulation of the gradient flow. In contrast to the EDB the *integrated evolutionary variational inequality* formulation is used in [Mie15] to pass to the limit. Methods relying on a time discretization, i.e., the minimizing movements scheme are employed in [Bra14, ABZ16, BCGS16]. However, EDP-convergence with tilting is based on the EDB.

In the sequel, we introduce the most important variational concepts within the realm of this thesis like Γ-convergence, gradient systems and EDP-convergence with tilting. Chapter 3 is concerned with the limit passage of a gradient system with a wiggly energy, which is published in [DFM18]. More precisely, Chapter 3 motivates the definition of EDP-convergence with tilting since it is shown that tilts are crucial to understand the limits of the kinetic relation. Moreover, we see clearly how the limiting dissipation potential depends on the wiggly part of the ε-dependent energy. Chapter 4 treats the limit passage of a diffusion equation on a domain with three thin layers by means of two different gradient systems. In the doubly linear, i.e., purely quadratic Hilbert-structure the limiting gradient system remains purely quadratic. More interestingly, when using the Boltzmann-Wasserstein gradient structure the ε-dependent dissipation potential is quadratic whereas the limiting dissipation potential is not, i.e., the type of dissipation changed. The passage to the membrane limit in Chapter 5 is also done via two different gradient structures. In both cases we have an ε-dependent quadratic dissipation potential and a non-quadratic effective dissipation potential.

1.1 Brief introduction to Γ-convergence

For a comprehensive introduction to Γ-convergence we refer to [Bra02, DM93]. The notion of Γ-convergence is especially suited for studing the convergence of Euler-Lagrange equations $0 = \mathrm{D}\mathbf{F}_\varepsilon(u_\varepsilon)$, i.e, if Γ-convergence $\mathbf{F}_\varepsilon \xrightarrow{\Gamma} \mathbf{F}_0$ holds then under suitable assumptions it follows that $u_\varepsilon \to u_0$ and $0 = \mathrm{D}\mathbf{F}_0(u_0)$. Hence, Γ-convergence lead to both, the limit of solutions and the limiting equation, which is also in the class of Euler-Lagrange equation. A similar concept is desirable for evolution equations, too.

Definition 1.1. *Let* X *be a Banach space and* τ *a topology on* X*, e.g. the strong or weak topology. Let* $\mathbf{F}_\varepsilon : \mathrm{X} \to (-\infty, \infty]$ *be a family of functionals. Then* $\mathbf{F}_\varepsilon \xrightarrow{\Gamma} \mathbf{F}_0$ *with respect to the topology* τ *if*

$$\forall\, u_\varepsilon \xrightarrow{\tau} u : \ \liminf_{\varepsilon\downarrow 0} \mathbf{F}_\varepsilon(u_\varepsilon) \geq \mathbf{F}_0(u), \tag{Γ-liminf}$$

$$\forall\, \hat{u}\, \exists\, \hat{u}_\varepsilon \xrightarrow{\tau} \hat{u} : \ \limsup_{\varepsilon\downarrow 0} \mathbf{F}_\varepsilon(\hat{u}_\varepsilon) \leq \mathbf{F}_0(\hat{u}). \tag{Γ-limsup}$$

Note that (Γ-limsup) is equivalent to $\forall\, \hat{u}\, \exists\, \hat{u}_\varepsilon \xrightarrow{\tau} \hat{u} : \ \lim \mathbf{F}_\varepsilon(\hat{u}_\varepsilon) = \mathbf{F}_0(\hat{u})$. However, in general it is more convenient to prove (Γ-limsup). We remark that it is sufficient for the proof of (Γ-liminf) to assume $\infty > \mathbf{F}_\varepsilon(u_\varepsilon)$. To proof (Γ-limsup) it is sufficient to consider $\hat{u} \in \mathrm{dom}(\mathbf{F}_0)$ only.

Lemma 1.2. *Let* τ *be induced by a metric* d_τ*. Then* (Γ-limsup) *is equivalent to* $\forall\, \hat{u}\, \forall\, n \in \mathbb{N}\, \exists (\hat{u}_\varepsilon^n)_{0<\varepsilon\leq 1}$ *such that* $\limsup_{\varepsilon\downarrow 0} \mathrm{d}_\tau(\hat{u}_\varepsilon^n, \hat{u}) = o_{n\uparrow\infty}(1)$ *and*

$$\limsup_{\varepsilon\downarrow 0} \mathbf{F}_\varepsilon(\hat{u}_\varepsilon^n) \leq \mathbf{F}_0(\hat{u}) + o_{n\uparrow\infty}(1). \tag{Γ-limsup'}$$

Proof. The implication (Γ-limsup)⇒(Γ-limsup') is trivial. Note that (Γ-limsup') reads

$$\forall\, n \in \mathbb{N}\, \exists\, \varepsilon_0(n)\, \forall\, \varepsilon \leq \varepsilon_0(n) : \ \mathbf{F}_0(\hat{u}) + \frac{1}{n}.$$

Without loss of generality we assume that $n \mapsto \varepsilon_0(n)$ is monotonously decreasing. Hence, it is easy to construct a monotonously increasing map $\varepsilon \mapsto n_\varepsilon$ satisfying $\varepsilon \geq \varepsilon_0(n_\varepsilon)$. Hence, $\limsup_{\varepsilon\downarrow 0} \mathbf{F}_\varepsilon(\hat{u}_\varepsilon^{n_\varepsilon}) \leq \mathbf{F}_0(\hat{u})$. □

In particular, it suffices to show (Γ-limsup) only on a dense subset (see [Bra02, Remark 1.29, Prop. 1.44]). Where the subset is dense in a topology σ

that is stronger than τ such that $\mathbf{F}_0$ is continuous, i.e., $(u^n \xrightarrow{\sigma} u) \Rightarrow (u^n \xrightarrow{\tau} u)$ and $(u^n \xrightarrow{\sigma} u) \Rightarrow \big(\mathbf{F}_0(u^n) \to \mathbf{F}_0(u)\big)$. In degenerate cases we may have that the ε-limit does not $\lim_{\varepsilon\downarrow 0} \hat{u}^n_\varepsilon = u$ depend on n. However, $\limsup_{\varepsilon\downarrow 0} \mathbf{F}_\varepsilon(\hat{u}^n_\varepsilon)$ may still depend on n, see e.g. [Bra02, Thm 6.4].

1.2 Introduction to gradient flows

When considering evolution equation originating from a physical system it is important to preserve physical properties, e.g., the second law of thermodynamics that is the mathematical entropy is decreasing along solutions. Hence, from a modeling perspective it is meaningful to use gradient flows (see e.g. [Pel14]). Several authors introduced gradient systems in various field, see e.g. [Ott01, JKO98, LM13, MM17, MPRT17, GM13, Mie11].

Definition 1.3. *We call a triple* $(\mathrm{X}, \mathcal{E}, \mathcal{R})$ *a generalized gradient system, where* X *is a Riemannian manifold,* $\mathcal{E} : [0,T] \times \mathrm{X} \to \mathbb{R} \cup \{\infty\} =: \overline{\mathbb{R}}$ *is called the energy and* $\mathcal{R} : \mathrm{TX} \to [0,\infty]$ *is called the dissipation potential with the tangent bundle* $\mathrm{TX} = \bigcup_{u\in \mathrm{X}} \{u\} \times \mathrm{T}_u\mathrm{X}$ *and for any* $u \in \mathrm{X}$ *we have*

$$\mathcal{R}(u, 0) = 0 \textit{ and } \mathrm{T}_u\mathrm{X} \ni v \mapsto \mathcal{R}(u, v) \textit{ is convex.}$$

Clearly, if X is a Banach space we have $\mathrm{TX} = \mathrm{X} \times \mathrm{X}$. The gradient system induces a gradient flow via the equation

$$-\,\mathrm{D}\mathcal{E}(t, u) \in \partial_{\dot u}\mathcal{R}(u, \dot u), \tag{1.1}$$

where $\partial_{\dot u}\mathcal{R}(u, \dot u) \subset \mathrm{T}^*_u\mathrm{X}$ denotes the convex subdifferential and $\mathrm{D}\mathcal{E}(u)$ is a suitable notion of differential of $\mathcal{E}$ giving the driving forces for the evolution.

The classic case is $\mathcal{R}(u, \dot u) = \frac{1}{2}\langle \mathbb{G}(u)\dot u, \dot u\rangle$ being quadratic and $\mathbb{G}(u) : \mathrm{T}_u\mathrm{X} \to \mathrm{T}^*_u\mathrm{X}$ being a state-dependent, symmetric, and positive semi definite operator. In this case, $\mathbb{G}(u)$ has to be seen as a Riemannian metric, whose inverse $\mathbb{K}(u) = \mathbb{G}(u)^{-1}$ gives the gradient of $\mathcal{E}$, namely,

$$\dot u = -\mathbb{K}(u)\mathrm{D}\mathcal{E}(u) = -\nabla_{\mathbb{K}}\mathcal{E}(u).$$

The operator $\mathbb{K}(u)$ is also called Onsager operator and defines a kinetic relation, i.e., a map from forces to velocities via $\xi \mapsto \mathbb{K}(u)\xi = v$. In theory of optimal transport this is called continuity equation.

For generalized gradient systems the Onsager operator is replaced by the subdifferential of the dual dissipation potential $\partial_\xi\mathcal{R}^*(u, \cdot)$ where $\mathcal{R}^* : \mathrm{T}^*\mathrm{X} \to [0,\infty]$ is the Legendre transform of $\mathcal{R}$, i.e.,

$$\mathcal{R}^*(u, \xi) = \sup_v \big\{\langle \xi, v\rangle - \mathcal{R}(u, v)\big\}.$$

Since $\mathcal{R}(u, v) \geq \mathcal{R}(u, 0) = 0$, we have $\mathcal{R}^*(u, \xi) \geq \mathcal{R}^*(u, 0) = 0$. By definition of the Legendre transform $\mathrm{T}^*_u\mathrm{X} \ni \xi \mapsto \mathcal{R}^*(u, \xi)$ is convex. It is well known and easy to check that in the quadratic case we have $\mathcal{R}^*(u,\xi) = \frac{1}{2}\langle \xi, \mathbb{K}(u)\xi\rangle$. By the properties of the Legendre transform ([Fen14]) the primal and the dual dissipation potential satisfy the Legendre–Fenchel equivalences, i.e.

$$\begin{aligned} &\text{(i) } \xi \in \partial_{\dot u}\mathcal{R}(u,\dot u) \quad \Longleftrightarrow \quad \text{(ii) } \dot u \in \partial_\xi \mathcal{R}^*(u,\xi) \\ &\Longleftrightarrow \quad \text{(iii) } \mathcal{R}(u,\dot u) + \mathcal{R}^*(u,\xi) = \langle \xi, \dot u\rangle. \end{aligned} \tag{1.2}$$

Note that (i) is a rate equation in TX, (ii) is a force balance in $\mathrm{T}^*\mathrm{X}$ also called Biot's equation, whereas (iii) is a power balance in $\mathbb{R}$.

1.2.1 Energy dissipation balance

Integrating (iii) with $\xi = -\mathrm{D}\mathcal{E}_t\big(u(t)\big)$ over time and assuming that the chain-rule

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}\big(t,u(t)\big) = \langle \mathrm{D}\mathcal{E}_t\big(u(t)\big), \dot u(t)\rangle + \int_0^T \partial_t\mathcal{E}_t\big(u(t)\big)\,\mathrm{d}t \quad \text{for a.a. } t\in(0,T) \tag{1.3}$$

holds we obtain equivalent to (1.1) the energy dissipation balance

$$\int_0^T \partial_t\mathcal{E}_t\big(u(t)\big)\,\mathrm{d}t + \mathcal{E}_0\big(u(0)\big) - \mathcal{E}_T\big(u(T)\big) = \int_0^T \Big\{\mathcal{R}(u,\dot u) + \mathcal{R}^*\big(u,-\mathrm{D}\mathcal{E}_t(u)\big)\Big\}\,\mathrm{d}t. \tag{EDB}$$

For notational convenience we abbreviate $\mathcal{E}_t : t \mapsto \mathcal{E}(t,\cdot)$. The *total dissipation functional*

$$\mathfrak{D}(u) := \int_0^T \Big\{\mathcal{R}(u,\dot u) + \mathcal{R}^*\big(u,-\mathrm{D}\mathcal{E}_t(u)\big)\Big\}\,\mathrm{d}t. \tag{1.4}$$

is a crucial quantity in the limit passage for gradient flows, as [Ser11, DL15, Mie16] and the Definition 1.10 show. Since it is always true that

$$\int_0^T \partial_t\mathcal{E}_t\big(u(t)\big)\,\mathrm{d}t + \mathcal{E}_0\big(u(0)\big) - \mathcal{E}_T\big(u(T)\big) \le \mathfrak{D}(u)$$

we may rewrite (EDB) as

$$\mathcal{E}_T\big(u(T)\big) + \mathfrak{D}(u) \le \mathcal{E}_0\big(u(0)\big) + \int_0^T \partial_t\mathcal{E}_t\big(u(t)\big)\,\mathrm{d}t.$$

Note that the main technical prerequisite is the validity of the chain rule (1.3) for all suitable curves $t\mapsto u(t)$. Often (geodesic) λ-convexity of the energy is used to show that the chain rule holds [AGS05, RMS08, MRS13].

Definition 1.4 (geodesic λ-convexity)**.** *Let* (X,d) *be a geodesic metric space. We say that a functional* $\mathcal{E}:\mathrm{X}\to(-\infty,\infty]$ *is geodesically* λ*-convex if for any* $u_0,u_1\in\mathrm{Dom}(\mathcal{E})$ *there exists a constant speed geodesic* γ*, i.e.,*

$$\forall\, t,s\in[0,1]: \quad \mathrm{d}\big(\gamma(t),\gamma(s)\big) = \mathrm{d}\big(\gamma(1),\gamma(0)\big)|t-s|$$

with $\gamma(0)=u_0, \gamma(1)=u_1$ *such that*

$$\forall\,\theta\in[0,1]: \quad \mathcal{E}\big(\gamma(\theta)\big) \le (1-\theta)\mathcal{E}\big(\gamma(0)\big) + \theta\mathcal{E}\big(\gamma(1)\big) - \frac{\lambda}{2}\theta(1-\theta)\mathrm{d}\big(\gamma(0),\gamma(1)\big)^2.$$

Moreover, if $\mathcal{E}$ is geodesic λ-convex then it is also geodesic Λ-convex for all $\Lambda < \lambda$. For Banach spaces geodesic curves connecting u_0 and u_1 are given by $s \mapsto (1-s)u_0 + s\ u_1 =: u_s$. Hence, λ-convexity holds if and only if

$$\forall\ \theta \in [0,1]: \quad \mathcal{E}(u_\theta) \le (1-\theta)\mathcal{E}(u_0) + \theta\mathcal{E}(u_1) - \frac{\lambda}{2}\theta(1-\theta)\|u_1 - u_0\|^2.$$

Note that for $\lambda = 0$ we have 0-convexity is the usual convexity.

Example 1.5. *With respect to the standard 2-Wasserstein metric* $\mathrm{d}_{\mathcal{W}_2}$ *both the Boltzmann entropy with respect to some equilibrium measure* π

$$\mathcal{E}_1(\mu) := \int \mathrm{E}_1(\mathrm{d}\mu/\mathrm{d}\pi)\,\mathrm{d}\pi \quad \textit{with} \quad \mathrm{E}_1(u) := u\log u - u + 1$$

and the Tsallis entropy

$$\mathcal{E}_m(\mu) := \int \mathrm{E}_m(\mathrm{d}\mu/\mathrm{d}\pi)\,\mathrm{d}\pi \quad \textit{with} \quad m > 1 \quad \textit{and} \quad \mathrm{E}_m(u) := \frac{u^m - u}{m-1} - u + 1$$

satisfy the chain rule and are convex (see [AGS05, Prop 9.3.2, Prop 10.3.18]).

1.2.2 Metric gradient flow formulation

For metric spaces there is a theory by its own for gradient flows, which are called curves of maximal slope [AGS05, RMS08]. We briefly mention the notion of curves of maximal slope and refer to the literature for details. We emphasize that (EDB) is equivalent to the metric formulation (1.5) stated below if the flow in the 2-Wasserstein space is with respect to the Boltzmann entropy or Tsallis entropy (see Section 2.1). In the metric setting the primal and dual dissipation potentials are replaced by metric quantities, the metric derivative and slope. The metric derivative is defined by

$$|u'|(t) := \lim_{h\to 0} \frac{\mathrm{d}\big(u(t), u(t+h)\big)}{h}$$

and the (local) slope is defined by

$$|\partial\mathcal{E}|(u) := \limsup_{v\to u} \frac{\big(\mathcal{E}(u) - \mathcal{E}(v)\big)^+}{\mathrm{d}(u,v)}.$$

As for (EDB) the validity chain rule is an elementary condition for the metric setting. In this case the chain rule reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}\big(t, u(t)\big) \ge \partial_t\mathcal{E}_t\big(u(t)\big) - |u'|(t)\cdot|\partial\mathcal{E}|\big(u(t)\big) \quad \text{for a.a. } t \in (0,T).$$

For a state independent dissipation potential $\Psi : \mathbb{R} \to [0,\infty]$ the metric formulation is

$$\int_0^T \partial_t\mathcal{E}_t\big(u(t)\big)\,\mathrm{d}t + \mathcal{E}_0\big(u(0)\big) - \mathcal{E}_T\big(u(T)\big) = \int_0^T \Big\{\Psi(|u'|) + \Psi^*\big(|\partial\mathcal{E}_t|(u)\big)\Big\}\,\mathrm{d}t. \qquad (1.5)$$

It is shown in the seminal work [JKO98] that the Fokker-Planck equation

$$\dot{\rho} = \operatorname{div}\big(\beta^{-1}\nabla\rho + \nabla\Phi(x)\rho\big)$$

is a gradient flow with respect to the Wasserstein metric and the free energy

$$\mathcal{E}(\rho) = \beta^{-1}\int_\Omega \mathrm{E}_1\left(\frac{\rho}{\mathrm{e}^{-\beta\Phi}}\right)\mathrm{e}^{-\beta\Phi}\,\mathrm{d}x.$$

1.3 EDP-convergence for gradient systems

We emphasize that there might exist several gradient structures inducing the same equation. In particular, the gradient structure gives additional information and has a physical meaning.

Example 1.6 ([DFM18]). *Let* $([0,\infty[,\mathbf{E}_j,\mathbf{R}_j)$ *for* $j\in\{1,2,3,4\}$ *with*

$$\begin{aligned}
\mathbf{E}_1(u) = \mathbf{E}_2(u) = \frac{1}{2}(1{-}u)^2, &\quad \mathbf{R}_1^*(\xi) = \frac{1}{2}\xi^2, \quad \mathbf{R}_2^*(u,\xi) = \frac{\frac{1}{2}\xi^2+\frac{1}{4}\xi^4}{1+(1{-}u)^2},\\
\mathbf{E}_3(u) = \mathbf{E}_4(u) = u\log u - u + 1, &\quad \mathbf{R}_3^*(u,\xi) = \frac{u{-}1}{2\log u}\xi^2,\\
&\quad \mathbf{R}_4^*(u,v) = 2\sqrt{u}\,\big(\cosh(\tfrac{1}{2}\xi)-1\big).
\end{aligned}$$

Then for all $j\in\{1,2,3,4\}$ *the gradient flow equation reads*

$$\dot{u} = 1-u.$$

There is also a close connection of gradient flows to theory of microscopic fluctuations in terms of the *large deviation principle* (see [MPR14]), i.e., the gradient systems arises from microscopic fluctuations by penalizing deviations from the value zero of the $\mathscr{L}$-function, where

$$\mathscr{L}(u,\dot{u}) = \mathcal{R}(u,\dot{u}) + \mathcal{R}^*\big(u,-\mathrm{D}\mathcal{E}(u)\big) + \langle\dot{u},\mathrm{D}\mathcal{E}(u)\rangle.$$

Note that under the assumption that the chain rule holds we have that

$$\int_0^T \mathscr{L}(u,\dot{u})\,\mathrm{d}t = 0$$

is exactly (EDB).

Considering the limit passage $\varepsilon\downarrow 0$ of gradient systems, we do not look only at solutions of the gradient flow. We look at fluctuations of solution, i.e., consider general curves $t\mapsto u_\varepsilon$. Similarly, Γ-convergence does not only consider solutions to the Euler-Lagrange equation. To gain an algorithm for the effective dissipation potential, we need to linearly decouple state and force, i.e.,

$$\xi = -\mathrm{D}\mathcal{E}_\varepsilon(u_\varepsilon) + \zeta.$$

In large deviation theory ζ is called *tilt* (see [Var16]). We emphasize that ζ shall not contain microscopic information, i.e., does not depend on ε. Introducing the tilt ζ corresponds to an external loading, i.e., to the tilted energy

$$\mathcal{E}_\varepsilon^\zeta(u) = \mathcal{E}_\varepsilon(u) - \langle \zeta, u \rangle.$$

The class of admissible tilts is specified for each problem.

Before giving the definition of *EDP-convergence with tilting* we introduce well-prepared E-convergence.

Definition 1.7 (pE-convergence, [Mie16]). *We say that* $(\mathrm{X}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ *pE-converges to* $(\mathrm{X}, \mathcal{E}_0, \mathcal{R}_{\text{eff}})$ *and write* $(\mathrm{X}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon) \xrightarrow{\text{pE}} (\mathrm{X}, \mathcal{E}_0, \mathcal{R}_{\text{eff}})$ *with respect to the topology* τ *if the conditions*

$$u_{0,\varepsilon} \xrightarrow{\tau} u_0 \quad \text{and} \quad \mathcal{E}_\varepsilon(0, u_{0,\varepsilon}) \to \mathcal{E}_0(0, u_0)$$

imply

$$\forall\, t > 0 : \quad u_\varepsilon(t) \xrightarrow{\tau} u(t) \quad \text{and} \quad \mathcal{E}_\varepsilon\big(t, u_\varepsilon(t)\big) \to \mathcal{E}_0\big(t, u(t)\big)$$

where u_ε *resp.* u *are the solutions to* (1.1) *induced by* $(\mathrm{X}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ *resp.* $(\mathrm{X}, \mathcal{E}_0, \mathcal{R}_{\text{eff}})$.

To ask for well-prepared initial conditions means that the sequence of initial conditions capture the microscopic feature of the energy, in other words the sequence of initial conditions recovers the limiting energy. If X is a Banach space and τ the weak topology we write $(\mathrm{X}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon) \overset{\text{pE}}{\rightharpoonup} (\mathrm{X}, \mathcal{E}_0, \mathcal{R}_{\text{eff}})$.

The well-preparedness of the initial conditions reflects that the gradient system is additional information for a given evolution. Different gradient system may have different microscopic features. Although it seems paradoxical it is not surprising that different gradient systems generating the same equation converge to different gradient systems that may induce different equations.

Example 1.8 ([Mie16, Corollary 3.8]). *Let* $a : \mathbb{R} \to (0, \infty)$ *be 1-periodic. The equation* $\dot\mu = -a_\varepsilon \mu$ *where* $a_\varepsilon : x \mapsto a(\varepsilon^{-1}x)$ *is induced by the gradient systems* $(\mathrm{M}_+(\overline{\Omega}), \mathcal{E}_\varepsilon^{(j)}, \mathcal{R}_\varepsilon^{(j)})$ *with*

$$\mathcal{E}_\varepsilon^{(1)}(\mu) = \int_{\overline{\Omega}} a_\varepsilon \, \mathrm{d}\mu, \qquad \mathcal{R}_\varepsilon^{(1)*}(\mu, \xi) = \frac{1}{2} \int_{\overline{\Omega}} \xi^2 \, \mathrm{d}\mu$$

$$\mathcal{E}_\varepsilon^{(2)}(\mu) = \int_{\overline{\Omega}} \frac{1}{a_\varepsilon} \, \mathrm{d}\mu, \qquad \mathcal{R}_\varepsilon^{(2)*}(\mu, \xi) = \frac{1}{2} \int_{\overline{\Omega}} a_\varepsilon^2 \xi^2 \, \mathrm{d}\mu.$$

Moreover, $(\mathrm{M}_+(\overline{\Omega}), \mathcal{E}_\varepsilon^{(j)}, \mathcal{R}_\varepsilon^{(j)}) \xrightarrow{\text{pE}} (\mathrm{X}, \mathcal{E}_0^{(j)}, \mathcal{R}_{\text{eff}}^{(j)})$ *with respect to the narrow topology where*

$$\mathcal{E}_0^{(1)}(\mu) = \int_{\overline{\Omega}} a_{\min} \, \mathrm{d}\mu, \qquad \mathcal{R}_{\text{eff}}^{(1)*}(\mu, \xi) = \frac{1}{2} \int_{\overline{\Omega}} .\xi^2 \, \mathrm{d}\mu$$

$$\mathcal{E}_0^{(2)}(\mu) = \int_{\overline{\Omega}} \frac{1}{a_{\max}} \, \mathrm{d}\mu, \qquad \mathcal{R}_{\text{eff}}^{(2)*}(\mu, \xi) = \frac{1}{2} \int_{\overline{\Omega}} a_{\max}^2 \xi^2 \, \mathrm{d}\mu$$

with $a_{\min} = \min_y a(y)$ *and* $a_{\max} = \max_y a(y)$. *However, the induced equations are different*

$$(\mathrm{M}_+(\overline{\Omega}), \mathcal{E}_0^{(1)}, \mathcal{R}_{\text{eff}}^{(1)}) : \quad \dot{\mu} = -a_{\min}\mu$$
$$(\mathrm{M}_+(\overline{\Omega}), \mathcal{E}_0^{(2)}, \mathcal{R}_{\text{eff}}^{(2)}) : \quad \dot{\mu} = -a_{\max}\mu.$$

Clearly, pE-convergence relates $\mathcal{E}_\varepsilon$ and $\mathcal{E}_0$ but does not give a relation between $\mathcal{R}_\varepsilon$ and $\mathcal{R}_{\text{eff}}$. It is the fundamental question of evolutionary Γ-convergence how $\mathcal{R}_{\text{eff}}$ can be derived from $\mathcal{R}_\varepsilon$. By assuming a certain type of convergence separately on $\mathcal{R}_\varepsilon$ and $\mathcal{R}_\varepsilon^*$ pE-convergence was proved in [SS04, Ser11]. For a state independent Hilbert space setting the relation between $\mathcal{R}_\varepsilon$ and $\mathcal{R}_{\text{eff}}$ is given by

$$\mathcal{R}_0(v) \le \liminf \mathcal{R}_\varepsilon(v_\varepsilon) \tag{1.6}$$
$$\text{and} \quad \mathcal{R}_0\big(-\mathrm{D}\mathcal{E}_0(u)\big) \le \liminf \mathcal{R}_\varepsilon\big(-\mathrm{D}\mathcal{E}_\varepsilon(u_\varepsilon)\big). \tag{1.7}$$

The condition (1.7) implicitly impose the closedness of the subdifferentials, which is essential for strong convergence notions for gradient system (see e.g. [Mie16]).

Definition 1.9 (Strong-weak closedness, Def 3.5 [Mie16]). *Let X be a reflexive Banach space. We say that the triples $(X, \mathcal{E}_\varepsilon, \partial\mathcal{E}_\varepsilon)_{\varepsilon\in[0,1]}$ satisfies the strong-weak closedness of the graph of $\partial\mathcal{E}_\varepsilon$, if the following holds:*

If $u_\varepsilon \to u$ in X, $\mathcal{E}_\varepsilon(u_\varepsilon) \to e_0$ in $\mathbb{R}$, $\xi_\varepsilon \in \partial\mathcal{E}_\varepsilon(u_\varepsilon)$ and $\xi_\varepsilon \rightharpoonup \xi$ in X^* then $\xi \in \partial\mathcal{E}_0(u)$ and $\mathcal{E}_0(u) = e_0$.

The definition of *EDP-convergence with tilting* is followed by its motivation.

Definition 1.10 (EDP-convergence with tilting). *Let $(\mathrm{X}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ be a sequence of gradient systems. For tilts $\zeta \in \mathbf{C} \subset \cap_{u\in\mathrm{X}}\mathrm{T}_u^*\mathrm{X}$ we define the tilted total dissipation functional*

$$\mathfrak{D}_\varepsilon^\zeta(u) := \int_0^T \Big\{ \mathcal{R}_\varepsilon(u, \dot{u}) + \mathcal{R}_\varepsilon^*\big(u, -\mathrm{D}\mathcal{E}_\varepsilon(u) + \zeta\big) \Big\} \,\mathrm{d}t. \tag{1.8}$$

Then we say that $(\mathrm{X}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ relaxed EDP-convergences to $(\mathrm{X}, \mathcal{E}_0, \mathcal{R}_{\text{eff}})$ and write $(\mathrm{X}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon) \xrightarrow{relEDP} (\mathrm{X}, \mathcal{E}_0, \mathcal{R}_{\text{eff}})$ with respect to τ if

(I) $(\mathrm{X}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon) \xrightarrow{pE} (\mathrm{X}, \mathcal{E}_0, \mathcal{R}_{\text{eff}})$ *with respect to* τ,

(II) $\mathfrak{D}_\varepsilon^\zeta \xrightarrow{\Gamma} \mathfrak{D}_\varepsilon^\zeta : u \mapsto \int_0^T \mathcal{M}_0(u, \dot{u}, -\mathrm{D}\mathcal{E}(u) + \zeta)\,\mathrm{d}t$ *with* $\mathcal{M}_0(u, v, \xi) \ge {}_{\mathrm{T}_u^*\mathrm{X}}\langle \xi, v\rangle_{\mathrm{T}_u\mathrm{X}}$ *and the contact set*

$$\mathbf{C}_{\mathcal{M}_0}(u) := \big\{ (v, \xi) \in \mathrm{T}_u\mathrm{X} \times \mathrm{T}_u^*\mathrm{X} \;\big|\; \mathcal{M}_0(u, v, \xi) = {}_{\mathrm{T}_u^*\mathrm{X}}\langle \xi, v\rangle_{\mathrm{T}_u\mathrm{X}} \big\} \tag{1.9}$$

is given by the graph of $\partial\mathcal{R}_{\text{eff}}(u, \cdot)$, i.e.,

(III) $\mathbf{C}_{\mathcal{M}_0}(u) := \big\{ (v, \xi) \in \mathrm{T}_u\mathrm{X} \times \mathrm{T}_u^*\mathrm{X} \;\big|\; \xi \in \partial\mathcal{R}_{\text{eff}}(u, v) \big\}$.

If $\mathcal{M}_0(u,v\xi)=\mathcal{R}_{\text{eff}}(u,v)+\mathcal{R}^*_{\text{eff}}(u,\xi)$, then we say that $(\mathrm{X},\mathcal{E}_\varepsilon,\mathcal{R}_\varepsilon)$ EDP-converges with tilting to $(\mathrm{X},\mathcal{E}_0,\mathcal{R}_{\text{eff}})$ and write $(\mathrm{X},\mathcal{E}_\varepsilon,\mathcal{R}_\varepsilon)\overset{\text{EDP-tilt}}{\longrightarrow}(\mathrm{X},\mathcal{E}_0,\mathcal{R}_{\text{eff}})$.

First of all, there are two main structural assumptions. First, we require the conservation of the Fenchel–Young estimate, i.e., $\mathcal{M}_0(u,v,\xi)\geq {}_{\mathrm{T}^*_u\mathrm{X}}\langle\xi,v\rangle_{\mathrm{T}_u\mathrm{X}}$, which preserves a form of energy dissipation balance. Second, the contact set is the graph of $\partial\mathcal{R}_{\text{eff}}$ which enables us to reformulate the evolution of $u:=\lim u_\varepsilon$ as a gradient flow.

By theory of Γ-convergence we can not expect that $\mathcal{M}_0$ is of $(\mathcal{R},\mathcal{R}^*)$ form like $\mathfrak{D}^\zeta_\varepsilon$. However, under not too restrictive assumptions ([DM93]) we may obtain the form

$$\mathfrak{D}^\zeta_0(u)=\int_0^T\mathcal{M}_0(u,\dot u,-\mathrm{D}\mathcal{E}_0(u)+\zeta)\,\mathrm{d}t.$$

By passing to the limit in (EDB)

$$\begin{gathered}\mathcal{E}_\varepsilon\big(u_\varepsilon(T)\big)+\mathfrak{D}_\varepsilon(u_\varepsilon)\leq\mathcal{E}_\varepsilon(u_{0,\varepsilon})\\ \downarrow\liminf\qquad\downarrow\liminf\qquad\downarrow\lim\\ \mathcal{E}_0\big(u(T)\big)+\mathfrak{D}_0(u)\leq\mathcal{E}_0(u_0)\end{gathered}$$

we arrive at

$$\begin{aligned}\mathcal{E}_0(u_0)&\overset{\text{(i)}}{=}\int_0^T{}_{\mathrm{T}^*_u\mathrm{X}}\langle-\mathrm{D}\mathcal{E}_0(u),\dot u\rangle_{\mathrm{T}_u\mathrm{X}}\,\mathrm{d}t+\mathcal{E}_0\big(u(T)\big)\\ &\overset{\text{(ii)}}{\leq}\int_0^T\mathcal{M}_0\big(u,\dot u,-\mathrm{D}\mathcal{E}_0(u)\big)\,\mathrm{d}t+\mathcal{E}_0\big(u(T)\big)\leq\mathcal{E}_0(u_0).\end{aligned}\tag{1.10}$$

Equality (i) holds by chain rule. Whereas estimate (ii) holds by the generalized Fenchel–Young estimate on $\mathcal{M}_0$. Moreover, we conclude equality in (1.10) and hence,

$$\mathcal{M}_0\big(u,\dot u,-\mathrm{D}\mathcal{E}_0(u)\big)={}_{\mathrm{T}^*_u\mathrm{X}}\langle-\mathrm{D}\mathcal{E}_0(u),\dot u\rangle_{\mathrm{T}_u\mathrm{X}}\quad\text{a.e. in }(0,T)$$

in other words, the limit evolution stays in the contact set of $\mathcal{M}_0$, i.e.,

$$\big(u,-\mathrm{D}\mathcal{E}_0(u)\big)\in\mathbf{C}_{\mathcal{M}_0}(u)\quad\text{a.e. in }(0,T).$$

By virtue of (III) in Definition 1.10 we can rewrite $\mathcal{M}_0$ on the contact set in terms of $\mathcal{R}_{\text{eff}}$ using the Legendre–Fenchel equivalences (1.2). Hence, we obtain

$$\begin{aligned}&\mathcal{E}_0(u_0)=\int_0^T\mathcal{M}_0\big(u,\dot u,-\mathrm{D}\mathcal{E}_0(u)\big)\,\mathrm{d}t+\mathcal{E}_0\big(u(T)\big)\\ \Longleftrightarrow&\\ &\mathcal{E}_0(u_0)=\int_0^T\mathcal{R}_{\text{eff}}(u,\dot u)+\mathcal{R}^*_{\text{eff}}\big(u,-\mathrm{D}\mathcal{E}_0(u)\big)\,\mathrm{d}t+\mathcal{E}_0\big(u(T)\big).\end{aligned}$$

As shown for the wiggly energy model (see Chapter 3) we may have the representation

$$\mathcal{M}_0\big(u,\dot u,-\mathrm{D}\mathcal{E}_0(u)\big)=\mathcal{R}_0(u,\dot u)+\mathcal{R}_0^*\big(u,-\mathrm{D}\mathcal{E}_0(u)\big). \tag{1.11}$$

But this relation holds only for the equilibrium driving force $-\mathrm{D}\mathcal{E}_0(u)$ and fails for tilted driving forces $-\mathrm{D}\mathcal{E}_0(u)+\zeta$. Hence, the underlying map from forces to velocities (Onsager operator/kinetic relation) $\xi\mapsto\partial\mathcal{R}_0^*(u,\xi)$ gives a different relation between forces and velocities. However, if we have *EDP-converges with tilting* then (1.11) holds also for the tilted driving forces.

hence, *relaxed EDP-convergence* is a notion of convergence for the kinetic relation. However, it is expected that the limiting kinetic relation is also given by a subdifferential of a dissipation potential.

2 Gradient flows in Wasserstein space

Since [JKO98] it is known that the diffusion equation is a Wasserstein gradient flow. Simiarly the porous medium equation can be considered as a Wasserstein gradient flow (see [Ott01]). The following two sections serve as a brief introduction to the Wasserstein space and common methods for calculus in Wasserstein spaces.

2.1 Introduction to Wasserstein metric

We equip the closure of a convex bounded domain $\Omega \subset \mathbb{R}^d$ with a metric induced by an elliptic coefficient A via

$$d(x_0, x_1) = \inf \left\{ \int_0^1 \sqrt{\dot{x}(s) \cdot \mathrm{A}^{-1}\big(x(s)\big)\dot{x}(s)}\,\mathrm{d}s : x(j) = x_j, j \in \{0,1\} \right\}$$

with $x \mapsto \xi \cdot \mathrm{A}^{-1}(x)\xi$ is l.s.c. for all $\xi \in \mathbb{R}^d$. In Chapter 4 we have a thin domain $\Omega_\varepsilon = \Sigma \times I_\varepsilon$ with $I_\varepsilon = (-\varepsilon - \varepsilon^\delta/2, \varepsilon + \varepsilon^\delta/2)$. For $x \in \Omega_\varepsilon$ we choose the natural decomposition $x = (y, z)$ with $y \in \Sigma$ and $z \in I_\varepsilon$. Then $\mathrm{A}(x) = a(z)\mathrm{I}_d$ with

$$a_\varepsilon(z) = \begin{cases} \varepsilon^{(2+\delta)} & \text{if } |z| < \varepsilon^\delta/2, \\ 1 & \text{else.} \end{cases}$$

Hence, the metric reads

$$d_\varepsilon(x_0, x_1) = \inf \left\{ \int_0^1 \frac{1}{\sqrt{a_\varepsilon(z(s))}} \|\dot{x}(s)\|\,\mathrm{d}s : x(j) = x_j, j \in \{0,1\} \right\}$$

Note that straight lines $s \mapsto (1-s)x_0 + sx_1$ are not necessarily geodesic curves. Due to generalization of the Benamou-Brenier formulation [BB00] the Wasserstein metric can be defined as follows

$$\mathcal{W}_2^2(\mu_0, \mu_1) = \inf \left\{ \int_0^1 \int_{\Omega_\varepsilon} a_\varepsilon |\mathbf{v}_t|^2 \,\mathrm{d}\mu_t \,\mathrm{d}t \;\middle|\; \dot{\mu}_t + \mathrm{div}(a_\varepsilon \mathbf{v}_t \mu_t) = 0, \mu_{|t=j} = \mu_j, j \in \{0,1\} \right\}.$$

The 2-Wasserstein space with respect to the metric d_ε is the space of probability measures equipped with the metric $\mathcal{W}_2$ denoted by $(\mathcal{P}_2(\Omega_\varepsilon), \mathcal{W}_2)$. In particular, the tangent space $\mathrm{T}_\mu \mathcal{P}_2(\Omega_\varepsilon)$ is isomorphic to $\big\{\mathrm{div}(a_\varepsilon \mathbf{v}\mu) | \mathbf{v} \in \mathrm{L}^2(\mathrm{d}\mu; \Omega_\varepsilon)\big\}$, with

$$\mathrm{L}^2(\mathrm{d}\mu; \Omega_\varepsilon) = \left\{ f : \int_{\Omega_\varepsilon} |f|^2 \,\mathrm{d}\mu < \infty \right\}.$$

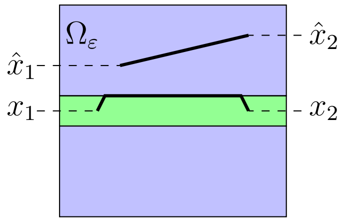

Figure 2.1: Depiction of geodesic curves.

We denote with $\mathrm{H}(\mathrm{d}\mu, \Omega_\varepsilon)$ the closure of $\{\nabla\varphi : \varphi \in \mathrm{C}^1(\Omega_\varepsilon)\}$ with respect to the $\mathrm{L}^2(\mathrm{d}\mu; \Omega_\varepsilon)$-norm. Note that for a fixed absolutely continuous curve $t \mapsto \mu_t$ the solution in the distributional sense to the continuity equation

$$\dot{\mu}_t + \mathrm{div}(a_\varepsilon \mathbf{v}_t \mu_t) = 0 \tag{CE}$$

is unique and satisfies $\mathbf{v} \in \mathrm{L}^2\big(0, T; \mathrm{H}(\mathrm{d}\mu_{\cdot_t}, \Omega_\varepsilon)\big)$. Note that the continuity can be interpreted as the Onsager operator/kinetic relation that maps forces to velocities, i.e., $\mathbb{K}(\mu)\xi = -\mathrm{div}(a_\varepsilon \nabla \xi\ \mu)$. Throughout this thesis we denote the time derivatives of μ by $\dot{\mu}$.

By virtue of the Benamou-Brenier formulation we have that the dissipation potential

$$\mathcal{R}_\varepsilon(\mu, \dot{\mu}) = \int_{\Omega_\varepsilon} \frac{a_\varepsilon(x)}{2} |\mathbf{v}|^2 \,\mathrm{d}\mu,$$

with $\mathbf{v} \in \mathrm{H}(\mathrm{d}\mu, \Omega_\varepsilon)$ satisfying (CE), is indeed given by the metric derivative [Lis06, Thm 3.10], i.e.

$$\mathcal{R}_\varepsilon(\mu, \dot{\mu}) = \frac{1}{2}|\mu'|^2.$$

Moreover, g given via

$$g^2(\mu) := \int_{\Omega_\varepsilon} a_\varepsilon(x) |\nabla \mathrm{E}'_m(\mu)|^2 \,\mathrm{d}\mu$$

is an upper gradient and satisfies $g \leq |\partial \mathcal{E}_m|(\mu)$ (see [Lis06, Lemma 4.3]).

In Subsection 1.2.2 it is shown that the metric flow is defined via the metric derivative and the slope.

Note that although the Wasserstein theory is done on whole $\mathbb{R}^d$, the results can be obtained via the trivial extension, i.e., $\widehat{\mu}(\mathbb{R}^d \setminus \Omega_\varepsilon) = 0$ and $\widehat{\mu}_{|\Omega_\varepsilon} = \mu$. In [Lis06, Subsubsection 5.2.2, Prop 5.9] it is shown that via the potential

$$V(x) = \begin{cases} 0 & \text{if } x \in \overline{\Omega}_\varepsilon, \\ +\infty & \text{if } x \in \mathbb{R}^d \setminus \overline{\Omega}_\varepsilon, \end{cases}$$

diffusion equations with homogenous Neumann boundary conditions can be formulated within the Wasserstein framework even without convexity assumptions on Ω_ε.

Similarly, we define a Wasserstein space with a nonlinear mobility $\mathsf{m}(u) = u^\gamma$ with $\gamma \in [0, 1)$ on $\mathcal{P}(\overline{\Omega}_\varepsilon)$. Here a reference probability measure π plays a central role.

$$\mathcal{W}_{\mathsf{m},2}^2(\mu_0,\mu_1) = \inf\left\{ \int_0^1 \int_{\Omega_\varepsilon} a_\varepsilon \left|\frac{\mathrm{d}\boldsymbol{\nu}_t}{\mathrm{d}\pi}\right|^2 \frac{1}{\mathsf{m}(u_t)}\,\mathrm{d}\pi\,\mathrm{d}t \;:\; \begin{array}{l} \dot\mu_t + \operatorname{div}(a_\varepsilon \boldsymbol{\nu}_t) = 0, \\ \mu_{|t=j} = \mu_j, j \in \{0,1\} \end{array} \right\}$$

with $\mu_t = u_t\pi + \pi^\perp$. Note that if we have for all $t \in [0,1]$ the absolute continuity $\mu_t = u_t\pi \ll \pi$ then $\boldsymbol{\nu}_t \ll \pi$ and

$$\mathbf{v}_t := \frac{\mathrm{d}\boldsymbol{\nu}_t}{\mathrm{d}\pi}\frac{1}{\mathsf{m}(u_t)} \quad \text{solves} \quad \dot\mu_t + \operatorname{div}(a_\varepsilon \mathbf{v}_t \mathsf{m}(u_t)\pi) = 0 \tag{CE$_\mathsf{m}$}$$

and

$$\mathcal{W}_{\mathsf{m},2}^2(\mu_0,\mu_1) = \inf\left\{ \int_0^1 \int_{\Omega_\varepsilon} a_\varepsilon |\mathbf{v}_t|^2 \mathsf{m}(u_t)\,\mathrm{d}\pi\,\mathrm{d}t \,\middle|\, (\mu,\mathbf{v}) \in (\text{CE}_\mathsf{m}), \mu_{|t=j} = \mu_j, j \in \{0,1\} \right\}.$$

We note that $\mathcal{W}_{\mathsf{m},2}$ may take the value $+\infty$. Thus we call $\mathcal{W}_{\mathsf{m},2}$ an extended metric. However, in the setting of Section 5.1 $\mathcal{W}_{\mathsf{m},2}$ is a metric (see [CLSS10, Thm 3.7]). By [CLSS10, Thm 4.12] we have that

$$\mathcal{E}_m(\mu) = \int_{\Omega_\varepsilon} \mathrm{E}_m\left(\frac{\mathrm{d}\mu}{\mathrm{d}\pi}\right)\mathrm{d}\pi$$

is geodesically convex for m,m and Ω_ε as in Section 5.1 but with $A = I$. However, by the methods employed in [Lis09] and [CLSS10] we expect that

$$g^2(\mu) := \int_{\Omega_\varepsilon} a_\varepsilon(x) |\nabla \mathrm{E}_m'(\mu)|^2 \mathsf{m}(u)\,\mathrm{d}\pi$$

is an upper gradient and satisfies $g \leq |\partial\mathcal{E}_m|(\mu)$. Hence, the porous medium equation is a curve of maximal slope with respect to the extended metric $\mathcal{W}_{\mathsf{m},2}$ and the energy $\mathcal{E}_m$.

2.2 Common methods

In the sequel we present methods that are used in sections 4.2, 5.1 and 5.2. In particular, we present methods for Wasserstein spaces from [Lis09] and [AGS05].

A common feature for four gradient systems considered here in this thesis is a vanishing middle layer, i.e., we have an Interval

$$I_\varepsilon =](1+\varepsilon)z_+, \varepsilon z_+[\cup [\varepsilon z_+, \varepsilon z_-] \cup]\varepsilon z_-, (1+\varepsilon)z_-[=: I_\varepsilon^+ \cup \overline{I}_\varepsilon^0 \cup I_\varepsilon^-$$

where $z_- < z_+$. For $\boldsymbol{\mu}_\varepsilon \in \mathcal{P}(I_\varepsilon)$, a probability measure we obtain after rescaling $\Phi_\varepsilon : I_\varepsilon \to I_1$ the push-forward measure $(\Phi_\varepsilon)_\# \boldsymbol{\mu}_\varepsilon = \mu_\varepsilon \in \mathcal{P}(I_1)$. For an absolutely continuous measure, i.e., $\mathrm{d}\boldsymbol{\mu}_\varepsilon = \mathbf{u}\,\mathrm{d}\hat{x}$ the density transforms to

$$\mathbf{m}_\varepsilon \mathbf{u} \circ \Phi_\varepsilon^{-1} =: \mathbf{m}_\varepsilon u_\varepsilon \quad \text{where} \quad \mathbf{m}_\varepsilon = \begin{cases} 1 & \text{on } I \setminus I^0 \\ \varepsilon & \text{on } I^0. \end{cases} \tag{2.1}$$

Clearly, we can control $\mathrm{m}_\varepsilon u_\varepsilon$ by the mass constraint. But additionally, we want to control u_ε on the middle layer. Therefore, we use the following estimate for any $v \in \mathrm{C}^1(I)$. We obtain for any $\widehat{z} \in I^0$ and any $z_1 \in I^\pm$

$$|v(\widehat{z})| = \Big| \int_{z_1}^{\widehat{z}} \partial_z v \,\mathrm{d}z + v(z_1) \Big| \leq \int_{I^0 \cup I^\pm} |\partial_z v| \,\mathrm{d}z + |v(z_1)|.$$

Integration over $z_1 \in I^\pm$ gives

$$|v(\widehat{z})| \leq \int_{I^0} |\partial_z v| \,\mathrm{d}z + \int_{I^\pm} \big(|\partial_z v| + |v|\big) \,\mathrm{d}z. \tag{2.2}$$

This observation is used if we have a bound on the dual dissipation potential of Wasserstein-type, i.e,

$$\sup_n \int_0^T \int_{I_1} |\partial_z \mathrm{E}'(u_n)|^2 u_n \,\mathrm{d}x \,\mathrm{d}t < \infty. \tag{2.3}$$

The following reasoning is suited to derive a priori bounds and convergence results from the bound (2.3).

As in [Lis09, p. 28] we conclude equi-integrability of $\partial_z \mathrm{E}'(u_n) u_n$. In fact, we even have equi-integrability for $|\partial_z \mathrm{E}'(u_n)|^p u_n$ for any $1 \leq p < 2$. In the case of the Boltzmann entropy $\mathrm{E}(u) = \mathrm{E}_1(u) = u \log u - u + 1$ we obtain weak compactness of $\partial_z \mathrm{E}_1'(u_n) u_n = \partial_z u_n$. In general, for equi-integrable $\{v_n\}$ with $0 \leq v_n$ we have equi-integrability of $w_n v_n$ if $\sup\limits_n \int |w_n|^p v_n \,\mathrm{d}x < \infty$. Applying Jensen's estimate for $\mathrm{d}\mathbb{P} = \frac{1}{\int v_n \,\mathrm{d}z} v_n \,\mathrm{d}z$ we estimate for conjugate exponents p and q and all measurable $B \subset [0,T] \times \overline{I}_1$

$$\int_B |w_n| v_n \,\mathrm{d}z \leq \left(\int_B v_n \,\mathrm{d}z \right)^{1/q} \left(\int_B |w_n|^p v_n \,\mathrm{d}z \right)^{1/p}. \tag{2.4}$$

Hence, by Dunford-Pettis ([DU77, Section III.2, Thm 15]), we have that $|w_n| v_n$ is relatively weakly compact in L^1.

Combining estimates (2.2) and (2.4) for the Boltzmann entropy we obtain the bound $\sup\limits_n \int_0^T \int_{I_1^0} |\partial_z u_n| + |u_n| \,\mathrm{d}z \,\mathrm{d}t < \infty$ only. Thus we obtain weak* compactness of $\partial_z u_n$ and u_n in the space of measures $\mathrm{Meas}\big([0,T] \times \overline{I}_1^0\big)$. Hence, up to a subsequence, $\partial_z u_n$ resp. u_n weak* converges to η resp. μ. Using the dual part of the total dissipation, we are able to conclude that $\eta \ll \mu$.

Lemma 2.1. *Let $u_n \rightharpoonup^* \mu$ and $\partial_z u_n \rightharpoonup^* \eta$ in $\mathrm{Meas}\big([0,T] \times \overline{I}_1^0\big)$ satisfying*

$$\sup_n \int_{[0,T] \times \overline{I}_1^0} \left| \frac{\partial_z u_n}{u_n} \right|^2 u_n \,\mathrm{d}(z,t) < \infty. \tag{2.5}$$

Then $\eta \ll \mu$ and the following limfinf estimate holds

$$\liminf \int_{[0,T] \times \overline{I}_1^0} \left| \frac{\partial_z u_n}{u_n} \right|^2 u_n \,\mathrm{d}(z,t) \geq \int_{[0,T] \times \overline{I}_1^0} \left| \frac{\mathrm{d}\eta}{\mathrm{d}\mu} \right|^2 \mathrm{d}\mu.$$

Proof. We estimate

$$\liminf \int_{[0,T]\times\overline{I}_1^0} \left|\frac{\partial_z u_n}{u_n}\right|^2 u_n \,\mathrm{d}(z,t) \geq \lim \int_{[0,T]\times\overline{I}_1^0} 2B|\partial_z u_n| - B^2 u_n \,\mathrm{d}(z,t)$$
$$\int_{[0,T]\times\overline{I}_1^0} 2B\,\mathrm{d}|\eta| - \int_{[0,T]\times\overline{I}_1^0} B^2\,\mathrm{d}\mu.$$

for any $B \in \mathrm{C}^0([0,T]\times\overline{I}_1^0)$. Note that $|\partial_z u_n| \rightharpoonup^* |\eta|$ where $|\eta| = \eta^+ + \eta^-$ is the Jordan decomposition of η. It remains to show, that $|\eta| \ll \mu$. Assume the contrary, i.e., there exists a measurable $\mathcal{A}$ such that $\mu(\mathcal{A}) = 0$ but $|\eta|(\mathcal{A}) > 0$. Then we choose a sequence B_k that concentrates on the set $\mathcal{A}$ and obtain

$$\liminf \int_{[0,T]\times\overline{I}_1^0} \left|\frac{\partial_z u_n}{u_n}\right|^2 u_n \,\mathrm{d}(z,t) \geq \int_{\mathcal{A}} 2B\,\mathrm{d}|\eta|.$$

Choosing $B \gg 1$ contradicts the assumption (2.5). Using the relation

$$\max_B \left\{2B\,\mathrm{d}\eta - B^2\,\mathrm{d}\mu\right\} = \left|\frac{\mathrm{d}\eta}{\mathrm{d}\mu}\right|^2 \mathrm{d}\mu$$

we finish the proof. □

We use the disintegration theorem ([DM88, 78 pp.] or for a statement of theorem with a notation closer to this thesis' one [AGS05, Thm 5.3.1]) we may decompose $\mathrm{d}\mu = \mathrm{d}\mu_t\,\mathrm{d}\mu^0$ where $\mathrm{d}\mu^0 : \mathcal{B}([0,T]) \ni \mathcal{A} \mapsto \mu(\mathcal{A}\times\overline{I}_1^0)$. For $\mathrm{d}\mu = u\,\mathrm{d}(z,t)$ we compute

$$\mathrm{d}\mu^0 = \left(\int_{\overline{I}_1^0} u(t,z)\,\mathrm{d}z\right)\mathrm{d}t \quad \text{and} \quad \mathrm{d}\mu_t = \frac{u(t,z)}{\int_{\overline{I}_1^0} u(t,z)\,\mathrm{d}z}\,\mathrm{d}z.$$

Let $f \in \mathrm{L}^1_\mu\big([0,T]\times\overline{I}_1^0\big)$ be the density of η with respect to μ, i.e., $\mathrm{d}\eta = f\,\mathrm{d}\mu$, then we can decompose η similarly. We easily see that $\mathrm{d}\eta = \mathrm{d}\mu_t'\,\mathrm{d}\mu^0$ where $\mathrm{d}\mu_t' = f(t,\cdot)\,\mathrm{d}\mu_t$. Using weak*-convergence we conclude that μ_t' and μ_t satisfy the following differential relation: $\forall\ \varphi \in \mathrm{C}^0\big([0,T]\times\overline{I}_1^0\big)$ with $\partial_z\varphi \in \mathrm{C}^0\big([0,T]\times\overline{I}_1^0\big)$ and $\varphi(\cdot,z^\pm) \equiv 0$ it holds

$$-\int_{[0,T]\times\overline{I}_1^0} \partial_z\varphi\,\mathrm{d}\mu_t\,\mathrm{d}\mu^0 = \int_{[0,T]\times\overline{I}_1^0} \varphi\,\mathrm{d}\mu_t'\,\mathrm{d}\mu^0.$$

Hence, by [AFP00, Thm 3.30] for μ^0-almost all $t \in [0,T]$ we may represent μ_t' as a derivative of a BV-function w_t on I^0 and conclude $\mathrm{d}\mu_t = w_t\,\mathrm{d}z$ as well as $f(t,\cdot)w_t = \partial_z w_t$. Since w_t has well-defined traces μ^0-a.e. we obtain

$$w_t(z_\pm)\,\mathrm{d}\mu^0 = u(t,z_\pm)\,\mathrm{d}t,$$

i.e, we derive the boundary conditions $w_t(z_\pm) = \frac{u(t,z_\pm)\,\mathrm{d}t}{\mathrm{d}\mu^0}$.

In order to treat the primal part that reads in the Wasserstein case

$$\mathfrak{D}_\varepsilon^{\mathrm{prim}}(\mu) = \int_0^T \int_{I_1} |\mathrm{v}|^2 u \,\mathrm{d}z\,\mathrm{d}t,$$

where $\mathrm{d}\mu = \mathbf{m}_\varepsilon u\,\mathrm{d}z$, with $\mathbf{m}_\varepsilon$ defined in (2.1), and v satisfies

$$\langle \xi, \dot\mu \rangle = \int_{I_1} \mathrm{v}\partial_z \xi \; u \,\mathrm{d}x,$$

we use the result [AGS05, Thm. 5.4.4] to pass to the limit in the kinetic relation.

Lemma 2.2 ([AGS05, Thm. 5.4.4]). *Let $\mu_n \rightharpoonup^* \mu$ and $v_n \in \mathrm{L}^2_{\mu_n}$ such that*

$$\sup_n \int |v_n|^2 \,\mathrm{d}\mu_n < \infty.$$

Then there exists $v \in \mathrm{L}^2_\mu$ and a subsequence such that

$$\forall\ \varphi \in \mathrm{C}_c^\infty : \quad \int v_n \varphi \,\mathrm{d}\mu_n \to \int v\varphi \,\mathrm{d}\mu.$$

Morover, for any convex $G : \mathbb{R} \to \mathbb{R}$ we have

$$\liminf_n \int G(v_n)\,\mathrm{d}\mu_n \geq \int G(v)\,\mathrm{d}\mu.$$

This result can be rewritten in the form, that η_n defined via $\mathrm{d}\eta_n = v_n\,\mathrm{d}\mu_n$ weak*-converges to η and the limit of v_n is given by $v = \frac{\mathrm{d}\eta}{\mathrm{d}\mu}$. Then, the liminf estimate is obtained as follows. For $G : \mathbb{R} \to \mathbb{R}$ convex we have also that $\mathbb{R}\times\mathbb{R}_+ \ni (a,b) \mapsto G\big(\frac{a}{b}\big)b$ is convex. Using Jensen's estimate we obtain

$$\int G(v_n)\,\mathrm{d}\mu_n = \int G\left(\frac{\mathrm{d}\eta_n}{\mathrm{d}\mu_n}\right)\mathrm{d}\mu_n \geq \sum_j \mu_n(A_j) G\left(\frac{\eta_n(A_j)}{\mu_n(A_j)}\right) \to \sum_j \mu(A_j) G\left(\frac{\eta(A_j)}{\mu(A_j)}\right).$$

giving

$$\liminf \int G(v_n)\,\mathrm{d}\mu_n \geq \int G(v)\,\mathrm{d}\mu.$$

In the Wasserstein space with nonlinear mobility we have the two objects $\mathrm{d}\mu_n^1 = u_n\,\mathrm{d}z$ and $\mathrm{d}\mu_n^2 = \mathsf{m}(u_n)\,\mathrm{d}z$. We can apply the above result only for $\mu_n^2 = \mu_n^2$. Hence, the limit identification $\mathrm{d}\mu = \mathsf{m}(u)\,\mathrm{d}z$ is needed to obtain a kinetic relation in terms of u, i.e.,

$$\langle \xi, \dot\mu^1 \rangle = \int_{I_1} \mathrm{v}\partial_z \xi \; \mathsf{m}(u)\,\mathrm{d}z.$$

3 Wiggly energy model

The results on the wiggly energy model studied in this chapter are published in [DFM18] and co-authored by Patrick Dondl and Alexander Mielke.

We apply the notion of relaxed EDP-convergence to the gradient flow

$$-\mathrm{D}\mathcal{E}_\varepsilon(t,u) = \partial\mathcal{R}(u,\dot u), \quad u(0) = u^0 \in \mathbb{R}, \tag{3.1}$$

where the wiggly energy has the form

$$\mathcal{E}_\varepsilon(t,u) = \Phi(u) - \ell(t)u + \varepsilon\kappa(u, \tfrac{1}{\varepsilon}u)$$

with a 1-periodic function $\kappa(u,\cdot)$ and $\mathcal{R}$ satisfies p-growth conditions and has a mild dependence on u. The limit passage of the equation for $2\mathcal{R}(u,v) = v^2$ is done in [Jam96, ACJ96] for a model explaining slip-stick motions in martensitic phase transformations is considered. Vector-valued versions (i.e. $u(t) \in \mathbb{R}^n$) of such gradient systems are considered in [Men02, Sul09].

In general, wiggly energy models result in a stick-slip motion due to the spatially rapidly varying energy landscape. Hence, the limit evolution cannot be a described as the gradient flow of the homogenized energy with the initial dissipation potential. In [ABZ16] the limit passage is performed using methods for gradient flows by means of the minimizing movement scheme. However, we prove relaxed EDP-convergence for this wiggly energy model and hence, give a relation between the microscopic and the effective dissipation potential. In particular, we find that the wiggly part κ of the energy enters the effective dissipation potential $\mathcal{R}_{\mathrm{eff}}$.

Under suitable assumptions it is well known from the above works (see e.g. [ACJ96, Men02, PT02, Sul09]) that the solutions u_ε of (3.1) converge for $\varepsilon \to 0$ to a limit u_0 that are solutions of the limiting gradient system $(\mathbb{R}, \mathcal{E}_0, \mathcal{R}_{\mathrm{eff}})$.

While the energy $\mathcal{E}_\varepsilon$ converges even uniformly to $\mathcal{E}_0 : (t,u) \mapsto \Phi(u) - \ell(t)u$, we lack the closedness of the subdifferentials (see Definition 1.9), which is essential for strong convergence notions for gradient system and is implicitly imposed in the Sandier-Serfaty approach [SS04] by the condition (1.7).

The Γ-limit of

$$\mathfrak{D}_\varepsilon^\zeta(u) = \int_0^T \mathcal{R}(u,\dot u) + \mathcal{R}^*\big(u, -\eta_\varepsilon - \partial_y\kappa(u,\varepsilon^{-1}u)\big)\,\mathrm{d}t$$

where $\eta_\varepsilon = \mathrm{D}\mathcal{E}_0(t,u) + \varepsilon\partial_u\kappa(u,\varepsilon^{-1}u) - \xi$ is computed by a generalization of classical homogenization tools ([Bra02, Thm. 3.1]). Therefore we introduce the functional $\mathfrak{J}_\varepsilon : \mathrm{W}^{1,p}(0,T) \times \mathrm{L}^{p'}(0,T) \ni (u,\xi) \to [0,\infty)$ with

$$\hat{\mathfrak{J}}_\varepsilon(u,\xi) = \int_0^T \mathcal{R}(u,\dot u) + \mathcal{R}^*\big(u, -\xi - \partial_y\kappa(u,\varepsilon^{-1}u)\big)\,\mathrm{d}t \tag{3.2}$$

and show that we can replace u_ε by w and η_ε by η, i.e, we can replace $\hat{\mathfrak{J}}_\varepsilon$ by

$$\mathfrak{J}_\varepsilon(u) = \int_0^T \mathcal{N}(t, \varepsilon^{-1}u, \dot{u})\,\mathrm{d}t$$

where

$$\mathcal{N}(t, y, v) = \mathcal{R}\big(w(t), v\big) + \mathcal{R}^*\big(w(t), -\eta(t) - \partial_y\kappa(w(t), y)\big)$$

and w is a placeholder for $u = \mathrm{w}\text{-}\lim u_\varepsilon$ and $\eta = \lim \eta_\varepsilon$.

In order to do so we state our assumptions on the dissipation potential $\mathcal{R}$ and on the wiggly part κ:

$$\mathcal{E}_\varepsilon(t,u) = \Phi(u) - \ell(t)u + \varepsilon\kappa(u, \tfrac{1}{\varepsilon}u) \text{ with } \Phi \in \mathrm{C}^1(\mathbb{R}),\ \ell \in \mathrm{C}^1([0,T]) \tag{3.3a}$$
$$\text{and } \kappa \in \mathrm{C}^1(\mathbb{R}^2) \text{ with } \kappa(u, y{+}1) = \kappa(u,y) \text{ for all } u, y \in \mathbb{R}; \tag{3.3b}$$
$$\mathcal{R} \in \mathrm{C}^1(\mathbb{R}^2), \quad \mathcal{R}(u,v) \geq 0, \quad \mathcal{R}(u,0) = 0; \tag{3.3c}$$
$$\forall\, u \in \mathbb{R}:\ \mathcal{R}(u,\cdot) \text{ is strictly convex}; \tag{3.3d}$$
$$\exists\, p \in\,]1,\infty[\ \exists\, c_1, c_2, c_3 > 0\ \exists \text{ modulus of continuity } \omega\ \ \forall\, u,\ \widehat{u},\ v \in \mathbb{R}:$$
$$c_1|v|^p {-} c_2 \leq \mathcal{R}(u,v) \leq c_3(1{+}|v|^p) \text{ and} \tag{3.3e}$$
$$|\mathcal{R}(u,v) {-} \mathcal{R}(\widehat{u},v)| \leq \omega(|u{-}\widehat{u}|)(1{+}|v|^p). \tag{3.3f}$$

The assumption on $\mathcal{R}$ are equivalent to imposing thes on $\mathcal{R}^*$, i.e, $\mathcal{R}^*$ satisfies

$$\mathcal{R}^* \in \mathrm{C}^1(\mathbb{R}^2), \quad \mathcal{R}^*(u,\xi) \geq 0, \quad \mathcal{R}^*(u,0) = 0; \tag{3.4a}$$
$$\forall\, u \in \mathbb{R}:\ \mathcal{R}^*(u,\cdot) \text{ is strictly convex}; \tag{3.4b}$$
$$\exists\, c_4, c_5, c_6 > 0\ \ \forall\, u,\ \widehat{u},\ \xi \in \mathbb{R}:$$
$$c_4|\xi|^{p'} {-} c_5 \leq \mathcal{R}^*(u,\xi) \leq c_6(1{+}|\xi|^{p'}) \text{ and} \tag{3.4c}$$
$$|\mathcal{R}^*(u,\xi) {-} \mathcal{R}^*(\widehat{u},\xi)| \leq C_p\omega(|u{-}\widehat{u}|)(1{+}|\xi|^{p'}), \tag{3.4d}$$

where $p' = p/(p-1)$ and $C_p > 1$ depending only on $p > 1$. It is a well known result from [Roc70, Thm 26.3] that strict convexity dualizes under the Legendre–Fenchel transform to differentiability and vice versa. The continuity (3.4d) follows from the estimate $|\partial\mathcal{R}^*(u,\xi)| \leq C_p'|\xi|^{p'-1}$.

Lemma 3.1. *Let $p > 1$ and $\mathcal{F} : X \to \mathbb{R}$ be convex and non-negative with at most p-growth, i.e., $0 \leq \mathcal{F}(v) \leq c + \|v\|^p$, then $\xi \in \partial\mathcal{F}(v)$ satisfies*

$$\|\xi\| \leq C_p(1 + \|v\|^{p-1})$$

with C_p dependend on $p > 1$.

Proof. We observe that $\mathcal{F}^*(\xi) \geq \hat{c}_p\|\xi\|^{p'} - c$ with $\hat{c}_p$ dependend on p and c from the assumption. Let $\xi \in \partial\mathcal{F}(v)$ then

$$c + 2^p\|v\|^p \geq \mathcal{F}(2v) \geq \mathcal{F}(v) + \langle\xi, v\rangle \geq 2\mathcal{F}(v) + \mathcal{F}^*(\xi) \geq \hat{c}_p\|\xi\|^{p'} - c.$$

Here we used that $\mathcal{F}(v) + \mathcal{F}^*(\xi) = \langle\xi, v\rangle$. By $p/p' = p - 1$ we conclude. $\square$

3.1 Main homogenization result

Throughout of this section we assume (3.3). Using these assumption we have indeed

Lemma 3.2. *$\exists\, C_{\hat{\mathcal{N}}} > 0$ such that $\hat{\mathcal{N}}(\xi, u, y, v) = \mathcal{R}(u, v) + \mathcal{R}^*\big(u, -\xi - \partial_y \kappa(u, y)\big)$ satisfies*

$$\big|\hat{\mathcal{N}}(\xi_1, u_1, y, v) - \hat{\mathcal{N}}(\xi_2, u_2, y, v)\big| \le C_{\hat{\mathcal{N}}}\Big(\omega(|u_1{-}u_2|)\big(1{+}|v|^p{+}|\xi_1|^{p'}{+}|\xi_2|^{p'}\big) \\ + (1{+}|\xi_1|^{p'-1}{+}|\xi_2|^{p'-1})|\xi_1{-}\xi_2|\Big).$$

Proof. We begin with

$$|\mathcal{R}^*(u_1, \eta_1) - \mathcal{R}^*(u_2, \eta_2)| = |\mathcal{R}^*(u_1, \eta_1) - \mathcal{R}^*(u_2, \eta_1)| + |\mathcal{R}^*(u_2, \eta_1) - \mathcal{R}^*(u_2, \eta_2)| \\ \le C_p \omega(|u_2 - u_1|)(1 + |\eta_1|^{p'}) + c(1 + |\eta_1|^{p'-1} + |\eta_2|^{p'-1})|\eta_1 - \eta_2|$$

where $\eta_j = \xi_j{+}\partial_y \kappa(u_j, y)$. The latter follows from the mean value theorem and the estimate $|\partial \mathcal{R}^*(u, \xi)| \le C_p'(1 + |\xi|^{p'-1})$. Using that $|\partial_y \kappa(u, y)|$ is bounded uniformly in u and y we estimate $|\eta_j|^q \le c_{q,\kappa}(1 + |\xi_j|^q)$ for $q \in \{p', p'-1\}$. The assumption (3.3f) gives the desired estimate. □

The basis of the proof of our homogenization result is [Bra02, Thm. 3.1]. So let us recall its statement.

Theorem 3.3 ([Bra02, Thm. 3.1])**.** *Let $1 < p < \infty$ and $g : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ be a borel function satisfying the growth condition*

$$\forall\, v \in \mathbb{R} :\ g(\cdot, v) \text{ is 1-periodic}, \qquad \forall\, y \in \mathbb{R} :\ g(y, \cdot) \text{ is convex}, \\ \exists\, p > 1\ \exists\, c_1, c_2, c_3 > 0\ \forall\, y, v \in \mathbb{R} : \quad c_1|v|^p - c_2 \le g(y, v) \le c_3\big(1 + |v|^p\big).$$

Then the functional $\mathfrak{J}_\varepsilon : \mathrm{W}^{1,p}(0, T) \ni q \mapsto \int_0^T g(\frac{1}{\varepsilon}u, \dot{u})\,\mathrm{d}t$ Γ-converges to the homogenized functional $\mathfrak{J}_0 : \mathrm{W}^{1,p}(0, T) \ni q \mapsto \int_0^T G_{\text{eff}}(\dot{u})\,\mathrm{d}t$, where G_{eff} is defined by

$$G_{\text{eff}}(V) := \lim_{L\to\infty}\ \inf\left\{\frac{1}{L}\int_0^L g\big(w(s){+}Vs, \dot{w}(s){+}V\big)\,\mathrm{d}s \big| w \in \mathrm{W}_0^{1,p}(0, L)\right\}$$

Note that we think of $g = \hat{\mathcal{N}}$ which gives additional properties and hence, a simplified representation of G_{eff}, which we present in the following proposition before stating and proving our homogenization result.

Proposition 3.4. *Consider a function $g \in \mathrm{C}(\mathbb{R}^2; [0, \infty[)$ with*

$$\forall\, v \in \mathbb{R} :\ g(\cdot, v) \text{ is 1-periodic}, \qquad \forall\, y \in \mathbb{R} :\ g(y, \cdot) \text{ is convex}, \tag{3.5a}$$
$$\exists\, p > 1\ \exists\, c_1, c_2 > 0\ \forall\, y, v \in \mathbb{R} : \quad c_1|v|^p - c_2 \le g(y, v) \le c_3\big(1 + |v|^p\big), \tag{3.5b}$$
$$\forall\, y \in \mathbb{R}\ \forall\, v \in \mathbb{R} \setminus \{0\} : \quad g(y, v) > g(y, 0) \ge 0. \tag{3.5c}$$

(A) For all $V \in \mathbb{R}$ we have the identity

$$\begin{aligned} G_{\text{eff}}(V) &:= \lim_{L\to\infty} \inf\Big\{ \frac{1}{L}\int_0^L g\big(w(s){+}Vs, \dot w(s){+}V\big)\,\mathrm{d}s \Big| w \in \mathrm{W}^{1,p}_{\text{per}}(0,L)\Big\} \\ &= \min\Big\{ \int_0^1 g\big(z(s), |V|\dot z(s)\big)\,\mathrm{d}s \Big| z \in \mathrm{W}^{1,p}(0,1),\ z(1) = z(0){+}\mathrm{sign}(V)\Big\}. \end{aligned} \tag{3.6}$$

(B) For $V \in \mathbb{R}$ we have the alternative characterization

$$G_{\text{eff}}(V) = \inf\Big\{ \int_0^1 g\big(y, \frac{V}{a(y)}\big)a(y)\,\mathrm{d}y \Big| a(y) > 0,\ \int_0^1 a(y)\,\mathrm{d}y = 1\Big\}, \tag{3.7}$$

and $V \mapsto G_{\text{eff}}(V)$ is continuous and convex.
(C) If g_1 and g_2 are functions satisfying (3.5) *with $g_j(y,v) \le c_3' + c_3|v|^p$ such that*

$$\exists\, \delta_1, \delta_2 > 0\ \forall\, y, v \in \mathbb{R}: \quad \big|g_1(y,v) - g_2(y,v)\big| \le \delta_1 + \delta_2|v|^p, \tag{3.8}$$

then the corresponding effective potentials $G^{(1)}_{\text{eff}}$ and $G^{(2)}_{\text{eff}}$ satisfy the estimate

$$\forall\, v \in \mathbb{R}: \quad \big|G^{(1)}_{\text{eff}}(v) - G^{(2)}_{\text{eff}}(v)\big| \ \le\ \delta_1 + \frac{\delta_2}{c_1}\big(c_1 + c_3' + c_3|v|^p\big) \tag{3.9}$$

where c_1 is from (3.5b). *Moreover, $G^{(j)}_{\text{eff}}$ hat p-growth, i.e.,*

$$c_1(|v|^p - 1) \le G^{(j)}_{\text{eff}}(v) \le c_3' + c_3|v|^p.$$

First, we remark that the additional constant c_3' introduced in (C) serves as a placeholder for $\|\eta\|^{p'}_{\mathrm{L}^{p'}}$.

Proof. We may rewrite the minimization in terms of $z(s) = w(s) + Vs$ as follows

$$G(L,V) := \inf\Big\{ \frac{1}{L}\int_0^L g\big(z(s), \dot z(s)\big)\,\mathrm{d}s \Big| z \in \mathrm{W}^{1,p}(0,L) \ \text{ with } \ z(L) = z(0) + VL\Big\} \tag{3.10}$$

and have to show $G(L,V) \to G_{\text{eff}}(V)$ as $L \to \infty$. For this we use the 1-periodicity of $g(\cdot, v)$. Moreover, we use the coercivity of g which guarantees the existence of minimizers such that the infimum $G(L,V)$ is attained.

We first treat the trivial case $V = 0$ and then $V > 0$. The case $V < 0$ is completely analogous to the case $V > 0$. The main argument for analyzing the minimizers in (3.10) is a simple cut-and-paste rearrangement of the graph $\{\, (s, z(s)) \mid s \in [0,L]\,\}$. On the one hand, we use the horizontal translation invariance, in particular, for

$$z_\theta(s) = \min\{1, |s-\theta|\},\ \theta \in [1, L-1]$$

we have $\int_0^L g(z_{\theta_1}, \dot z_{\theta_1})\,\mathrm{d}s = \int_0^L g(z_{\theta_2}, \dot z_{\theta_2})\,\mathrm{d}s$ since g does not depend on s. On the other hand, we use a vertical integer-translation invariance, i.e, for any z and

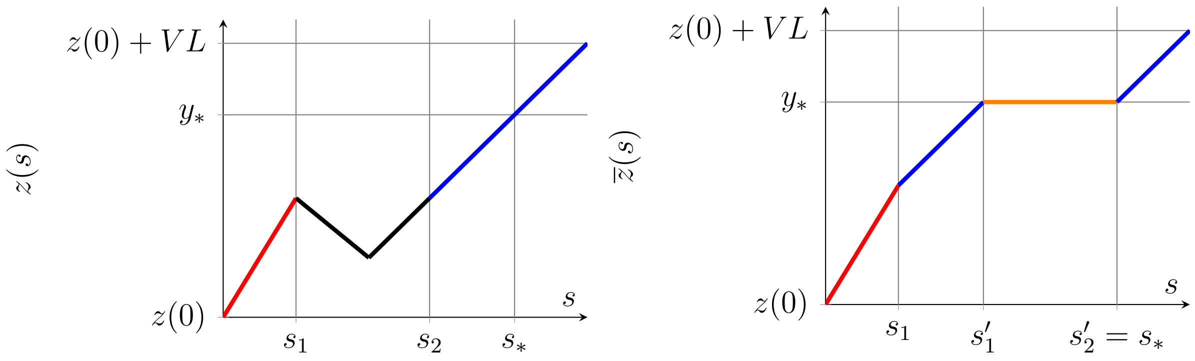

Figure 3.1: The new function $\overline{z}$ (right side) is constructed from the non-increasing function z (left side) by removing non-monotone part on $[s_1, s_2]$ and by inserting a flat part of the same length s_2-s_1 with value $\overline{z}(s) = y_* \in \operatorname{armgin} g(\cdot, 0)$.

any $k \in \mathbb{Z}$ we have with $z_k = z+k$ that $\int_0^L g(z, \dot{z})\,\mathrm{d}s = \int_0^L g(z_k, \dot{z}_k)\,\mathrm{d}s$ since $y \mapsto g(y, v)$ is 1-periodic for all v.

Step 1(A). The case $V = 0$.
We first observe that $G(L, 0) = g_{\min} := \min\{g(y, 0)|y \in \mathbb{R}\}$, since $g(y, v) \geq g_{\min}$ and we can choose $w \equiv y_*$ with $g(y_*, 0) = g_{\min}$. The minimizer z for (3.10) is given by $z \equiv y_*$.

Step 2(A). Monotonicity of z. Let $V > 0$, $LV \geq 1$ and z such that there exist s_1 and s_2 with $0 \leq s_1 < s_2 \leq L$ and $z(s_1) = z(s_2)$. We rearrange the graph of z such that $z'(s_1') = z'(s_2') = y_*$ with $s_2' - s_1' = s_2 - s_1$. Minimizing over $\overline{z}$ such that $\overline{z}_{[0,L]\setminus[s_1',s_2']} = z'_{[0,L]\setminus[s_1',s_2']}$ gives $\overline{z}_{[s_1',s_2']} = y_*$ with y_* from Step 1.

Using $LV \geq 1$ the intermediate-value theorem provides $s_* \in [0, L] \setminus]s_1, s_2[$ such that $z(s_*) = y_*$. For the case $s_* \geq s_2$ we obtain

$$\overline{z}(s) = \begin{cases} z(s) & \text{for } s \in [0, L] \setminus]s_1, s_*[, \\ z(s+s_2-s_1) & \text{for } s \in [s_1, s_1+s_*-s_2], \\ y_* & \text{for } [s_1+s_*-s_2, s_*]. \end{cases}$$

Here $s_1' = s_* - s_2 + s_1$ and $s_2' = s_*$. For an illustration of the rearrangement see Figure 3.1. The case $s_* \leq s_1$ is similar. Thus we estimate

$$\begin{aligned} \int_0^L g(z, \dot{z})\,\mathrm{d}s &= \int_{[0,L]\setminus]s_1',s_2'[} g(\overline{z}, \dot{\overline{z}})\,\mathrm{d}s + \int_{s_1}^{s_2} g(z, \dot{z})\,\mathrm{d}s \\ &\overset{\text{(i)}}{\geq} \int_{[0,L]\setminus]s_1',s_2'[} g(\overline{z}, \dot{\overline{z}})\,\mathrm{d}s + \int_{s_1'}^{s_2'} g(y_*, 0)\,\mathrm{d}s = \int_0^L g(\overline{z}, \dot{\overline{z}})\,\mathrm{d}s. \end{aligned}$$

where (i) is strict "$>$" if we do not have $z_{[s_1,s_2]} \in \operatorname{armgin} g(\cdot, 0)$. By construction we have $\overline{z} \in \mathrm{W}^{1,p}(0, L)$ and $\overline{z}(L) = \overline{z}(0)+LV$. Hence, $\overline{z}$ is a competitor for the minimization problem $G(L, V)$.

Step 3(A). $\forall V > 0 \; \forall k \in \mathbb{N}$ with $k/V \geq 1$ we have $G(k/V, V) = G(1/V, V)$.
We start from a minimizer w_V for $G(1/V, V)$ and use the 1-periodicity of $g(\cdot, v)$.

Extending w_V periodically to $w_V^k \in \mathrm{W}^{1,p}_{\mathrm{per}}(0,k/V)$ we can insert it as competitor for $G(k/V,V)$ and conclude $G(k/V,V) \le G(1/V,V)$.

For the opposite estimate we consider a fixed $k \ge 2$ and take a minimizer $w \in \mathrm{W}^{1,p}_{\mathrm{per}}(0,k/V)$ for $G(k/V,V)$. We extend w periodically to all of $\mathbb{R}$ and define $z : \mathbb{R} \ni s \mapsto w(s) + sV$ and

$$T := \{\, s_2{-}s_1 \mid s_1, s_2 \in \mathbb{R},\ z(s_2) = z(s_1) + 1 \,\} \quad \text{and } \tau_* := \min T.$$

The set T is non-empty as $z(k/V) = z(0) + k$. By Step 2 z is monotone, hence, $\tau_* > 0$. Choosing s_j with $z(s_j) = z(0) + j$ for $j = 1, ..., k{-}1$ and setting $s_0 = 0$ and $s_k = k/V$, we have $k/V = \sum_{j=1}^k (s_j{-}s_{j-1})$. Thus, at least one $s_j{-}s_{j-1}$ is less or equal $1/V$, which implies $\tau_* \le 1/V$.

By shifting z horizontally, we may assume $z(\tau_*) = z(0){+}1$. If $\tau_* = 1/V$ we have $z(1/V) = z(0) + 1$ so that $w : s \mapsto z(s) - Vs$ satisfies $w(0) = w(1/V) = w(k/V)$. hence, $w|_{[0,1/V]}$ is a competitor for $G(1/V,V)$, and $w_{[1/V,k/V]}$ is a competitor for $G((k{-}1)/V,V)$ (after shifting s to $s - 1/V$). Hence, we obtain

$$\begin{aligned} \frac{k}{V} G(k/V,V) &= \int_0^{1/V} g(w{+}Vs, \dot w{+}V)\,\mathrm{d}s + \int_{1/V}^{k/V} g(w{+}Vs, \dot w{+}V)\,\mathrm{d}s \\ &\ge \frac{1}{V} G(1/V,V) + \frac{k{-}1}{V} G((k{-}1)/V,V). \end{aligned} \tag{3.11}$$

We want to show the same lower bound for the case $\tau_* < 1/V$. This is done by a cut-and-paste rearrangement. We decompose $[0,k/V]$ into at most 5 parts via $0 < t_1 < t_2 < t_3 < t_4 \le k/V$. We set $t_2 := \tau_* < t_3 := 1/V$ and choose $t_4 > 1/V$ such that $z(t_4) = z(0) + j_*$ with $j_* \ge 2$ and $z(t_4{-}t_3) \ge z(0) + j_* - 1$. Now the intermediate-value theorem applied to the difference of $z|_{0,\tau_*}$ and $\overline{z} : [0,\tau_*] \ni s \mapsto z(t_4{-}t_3{+}s) - j_* {+} 1$ gives at least one zero $t_1 \in [0,\tau_*]$ as $z(0) \le \overline{z}(0) = z(t_4{-}t_3) - j_* {+} 1$ and $\overline{z}(t_3) = z(\tau_*) \le z(t_3)$ by monotonicity.

We define the rearrangement $\widehat{z}$ as a concatenation of vertically shifted versions of z on the intervals $[0,t_1]$, $[t_3,t_4]$, $[t_2,t_3]$, $[t_1,t_2]$, and $[t_4,k/V]$, namely

$$\widehat{z}(s) = \begin{cases} z(s) & \text{for } s \in [0,t_1] \cup [t_4,k/V], \\ z(s{+}t_4{-}t_3) - j_* + 1 & \text{for } s \in [t_1,t_2'], \\ z(s{+}t_2{-}t_3) & \text{for } s \in [t_2',t_3'], \\ z(s{+}t_2{-}t_4) + j_* - 1 & \text{for } s \in [t_3',t_4], \end{cases}$$

where $t_2' = t_3$ and $t_3' = t_4 - t_2 + t_1$. See Figure 3.2 for an illustration.

By construction z and $\widehat{z}$ are minimizers for $G(k/V)$, but $\widehat{z}$ additionally satisfies $\widehat{z}(1/V) = \widehat{z}(0) + 1$, as in the case $\tau_* = 1/V$. By induction we conclude $G(k/V,V) \ge G(1/V,V)$. Since the opposite estimate was shown above, we obtain the result $G(k/V,V) = G(1/V,V)$.

Step 4(A). Limit $G(L,V) \to G(1/V,V)$ for $L \to \infty$.
We already know the values at $G(k/V,V) = G(1/V,V)$, and now estimate the function for $L \in\,]k/V,(k{+}1)/V[$. Using $g_V^* = \max\{\, g(u,V) \mid u \in \mathbb{R} \,\}$ and taking

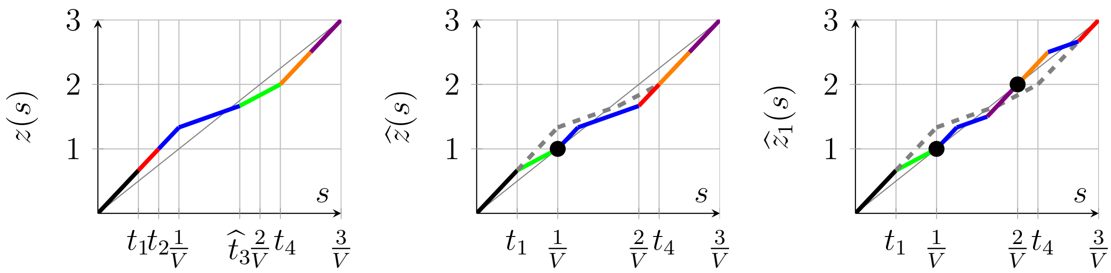

Figure 3.2: Rearrangement of z leads to $\widehat{z}$, which intersect the diagonal $s \mapsto z(0) + Vs$ at $s = 1/V$ (filled circle). With $\widehat{t}_3 = t_4 - t_3 + t_1$, the parts of the graph associated with $[t_1, t_2]$ and $[\widehat{t}_3, t_4]$ are interchanged by vertical integer-valued shifting and horizontal adjustment to make the function continuous.

the minimizer z_L for $G(L, V)$ we extend $z_L \in \mathrm{W}^{1,p}(0, L)$ to $\widetilde{z} \in \mathrm{W}^{1,p}(0, (k{+}1)/V)$ via $\widetilde{z}(s) = z(0) + sV$ for $s > L$, then

$$
\begin{aligned}
L\, G(L, V) = \int_0^L g(z_L, \dot{z}_L)\,\mathrm{d}s &\geq \int_0^{(k+1)/V} g(\widetilde{z}, \dot{\widetilde{z}})\,\mathrm{d}s - g_V^* \big(\frac{k{+}1}{V} - L\big) \\
&\geq \frac{k{+}1}{V} G((k{+}1)/V, V) - g_V^*/V \ \geq L\, G(1/V, V) - g_V^*/V.
\end{aligned}
$$

This implies $\liminf_{L\to\infty} G(L, V) \geq G(1/V, V)$. The opposite inequality follows by taking the minimizer $z_{k/V}$ and extending it affinely to a competitor for $G(L, V)$. This results in $\frac{k}{V} G(1/V, V) = \frac{k}{V} G(k/V, V) \geq LG(L, V) - g_V^*/V$. Hence, it follows that $\limsup_{L\to\infty} G(L, V) \leq G(1/V, V)$ and consequently $G(L, V) \to G(1/V, V)$ is established.

To establish the identity (3.6) we simply observe that the minimizers z of (3.6) and the minimizers w of $G(1/V, V)$ are related by $z(s) = w(|V|s) + \mathrm{sign}(V)\, s$. Thus, part (A) is established.

Step 5(B). Convexity of G_{eff}.
Let $V \neq 0$. Obviously monotone functions $s \mapsto z(s)$ as competitors in (3.6) can be approximated by strictly monotone functions in $\mathrm{W}^{1,p}(0, T)$. For these functions we can invert $y = z(s)$ to obtain $s = \sigma(y)$. Thus for $a(y) = \mathrm{sign}(V)\sigma'(y)$ we have $a(y) > 0$ and $\int_0^1 a(y)\,\mathrm{d}y = 1$. Thus, transforming the integral in (3.6) gives the desired formula (3.7). For $V = 0$ a dirac sequence concentrating on $\mathrm{argmin} g(y, 0)$ completes the proof of formula (3.7) for $V \in \mathbb{R}$.

The convexity of $g(y, \cdot)$ implies the convexity of $(v, a) \mapsto g(y, v/a)a =: h(y, a, v)$. With this we set $\mathcal{H}(a, v) = \int_0^1 h(y, a(y), v)\,\mathrm{d}y$, which is still convex in (a, v). Thus, for $\theta \in\,]0, 1[$ and $v_0, v_1 \in \mathbb{R}$ we choose for $\varepsilon > 0$ functions a_0 and a_1 such that

$\mathcal{H}(a_j, v_j) \leq G_{\text{eff}}(v_j) + \varepsilon$ for $j = 0$ and 1. For $v_\theta = (1-\theta)v_0 + \theta v_1$ we obtain

$$G_{\text{eff}}(v_\theta) = \inf\left\{\mathcal{H}(a, v_\theta) \,\Big|\, \int_0^1 a(y)\,\mathrm{d}y = 1\right\} \leq \mathcal{H}\big((1-\theta)a_0 + \theta a_1, v_\theta\big)$$
$$\overset{\mathcal{H}\text{ cvx}}{\leq} (1-\theta)\mathcal{H}(a_0, v_0) + \theta\mathcal{H}(a_1, v_1) \leq (1-\theta)G_{\text{eff}}(v_0) + \theta G_{\text{eff}}(v_1) + \varepsilon.$$

As $\varepsilon > 0$ was arbitrary the desired convexity is established.

Step 6(C). Continuous dependence of G_{eff} from g.
To obtain (3.9) we first consider the case $v_1 = v_2 = V$ and denote by z_j any minimizers for $G_j(1/V, V)$. By comparing with $\widehat{z}(s) = \text{sign}(V)s$ we first obtain the upper bound

$$G_{\text{eff}}^{(j)}(V) = G_j(1/V, V) = \int_0^1 g_j(z_j, |V|\dot{z}_j)\,\mathrm{d}s \leq \int_0^1 g_j(s, |V|)\,\mathrm{d}s \leq c_3' + c_3|V|^p.$$

Second, using the lower bound for g_j we find

$$G_{\text{eff}}^{(j)}(V) = \int_0^1 g_j(z_j, |V|\dot{z}_j)\,\mathrm{d}s \geq c_1|V|^p \int_0^1 |\dot{z}_j|^p\,\mathrm{d}s - c_1,$$

which gives the a priori estimate $c_1|V|^p \int_0^1 |\dot{z}_j|^p\,\mathrm{d}s \leq c_1 + c_3' + c_3|V|^p$. Now we compare the two effective potentials as follows

$$\begin{aligned}
G_{\text{eff}}^{(2)}(V) - G_{\text{eff}}^{(1)}(V) &= \int_0^1 \big(g_2(z_2, |V|\dot{z}_2) - g_1(z_1, |V|\dot{z}_1)\big)\,\mathrm{d}s \\
&\leq \int_0^1 \big(g_2(z_1, |V|\dot{z}_1) - g_1(z_1, |V|\dot{z}_1)\big)\,\mathrm{d}s \;\leq\; \int_0^1 \big(\delta_1 + \delta_2|V|^p|\dot{z}_1|^p\big)\,\mathrm{d}s \\
&= \;\delta_1 + \delta_2|V|^p \int_0^1 |\dot{z}_1|^p\,\mathrm{d}s \;\leq\; \delta_1 + \frac{\delta_2}{c_1}\big(c_1 + c_3' + c_3|V|^p\big).
\end{aligned}$$

By interchanging 1 and 2, we obtain the same bound for $G_{\text{eff}}^{(1)}(V) - G_{\text{eff}}^{(2)}(V)$ and (3.9) is established. □

Combining Lemma 3.2 and Proposition 3.4(C) we obtain

Corollary 3.5. *Let* $g^{(t)}(y, v) = \hat{\mathcal{N}}(\eta(t), u(t), y, v)$ *and* $g(y, v) = \hat{\mathcal{N}}(⨍\eta, ⨍u, y, v)$ *with* $t \in [t_1, t_2]$ *and* $⨍ h := |t_2 - t_1|^{-1} \int_{t_1}^{t_2} h(t)\,\mathrm{d}t$*. Then*

$$\begin{aligned}
\int_{t_1}^{t_2} |G_{\text{eff}}^{(t)}(\dot{u}) - G_{\text{eff}}(\dot{u})|\,\mathrm{d}t \leq C_{\hat{\mathcal{N}}}\Big(&\sup_{t\in[t_1,t_2]} \omega(|u(t) - ⨍u|)(1 + 2\|\eta\|^p_{\mathrm{L}^p(t_1,t_2)}) \\
&+ (|t_2 - t_1|^{\frac{1}{p}} + 2\|\eta\|^{p'-1}_{\mathrm{L}^{p'}(t_1,t_2)})\|\eta - ⨍\eta\|_{\mathrm{L}^{p'}(t_1,t_2)}\Big) \\
+ \frac{C_{\hat{\mathcal{N}}}}{c_1} &\sup_{t\in[t_1,t_2]} \omega(|u(t) - ⨍u|)\big(c + \|\hat{\eta}\|^{p'}_{\mathrm{L}^{p'}(t_1,t_2)} + c_3\|\dot{u}\|^p_{\mathrm{L}^p(t_1,t_2)}\big).
\end{aligned}$$

Here $\hat{\eta}$ *is a majorant of* η *and* $⨍\eta$ *and* c *depends on* c_1 *from* (3.3e), c_6 *from* (3.4c) *and on* $\|\partial_y\kappa\|_\infty$.

The next step is to prove the Γ-liminf estimate for $\hat{\mathcal{J}}_\varepsilon$ as $u_\varepsilon \rightharpoonup u$ in $\mathrm{W}^{1,p}(0,T)$ and $\eta_\varepsilon \to \eta$ in $\mathrm{L}^{p'}(0,T)$, in short: $(u_\varepsilon, \eta_\varepsilon) \underset{\mathrm{w}\times\mathrm{s}}{\longrightarrow} (u,\eta)$. The key ingredients of the proof are Lemma 3.2,[Bra02, Thm 3.1], Proposition 3.4 and Corollary 3.5. Defining the contact potential via

$$\mathcal{M}(u,v,\eta) = G_{\text{eff}}(v) \quad \text{for} \quad g(y,v) = \mathcal{R}\big(u,v\big) + \mathcal{R}^*\big(u, -\eta - \partial_y\kappa(u,y)\big)$$

gives

$$\begin{aligned} &\mathcal{M}(u,v,\eta) = \inf_{z\in \mathrm{W}^{1,p}_v}\Big(\int_0^1 \big[\mathcal{R}(u,|v|\dot{z}(s)) + \mathcal{R}^*\big(u,\eta - \partial_y\kappa(u,z(s))\big)\big]\,\mathrm{d}s\Big), \\ &\text{where } \mathrm{W}^{1,p}_v = \big\{v\in \mathrm{W}^{1,p}(0,1)\big|z(1) = z(0)+\mathrm{sign}(v)\big\}. \end{aligned} \tag{3.12}$$

Finally, the Γ-liminf estimate reads

Proposition 3.6 (The liminf estimate). *Let* $\mathfrak{J}_\varepsilon : \mathrm{W}^{1,p}(0,T)\times \mathrm{L}^{p'}(0,T) \to \mathbb{R}$ *be defined as in* (3.2). *Then,*

$$(u_\varepsilon,\eta_\varepsilon) \underset{\mathrm{w}\times\mathrm{s}}{\longrightarrow} (u,\eta) \;\; \textit{in } \mathrm{W}^{1,p}(0,T)\times \mathrm{L}^{p'}(0,T) \quad \Longrightarrow \quad \mathfrak{J}_0(u,\eta) \le \liminf_{\varepsilon\searrow 0} \mathfrak{J}_\varepsilon(u_\varepsilon,\eta_\varepsilon)$$

where

$$\mathfrak{J}_0(u,\eta) = \int_0^T \mathcal{M}(u,\dot{u},\eta)\,\mathrm{d}t.$$

Proof. In order to generalize [Bra02, Thm 3.1] we carefully treat the additional time dependence $t\mapsto \hat{\mathcal{N}}(\eta_\varepsilon(t),u_\varepsilon(t),y,v)$. Therefore let $0=t_0<t_1<\ldots<t_N=T$ be an arbitrary partition. By Lemma 3.2 we estimate for $j\in\{1,\ldots,N\}$ and $t\in[t_{j-1},t_j]$ and $⨍_j h := |t_j - t_{j-1}|^{-1}\int_{t_{j-1}}^{t_j} h(t)\,\mathrm{d}t$

$$\begin{aligned} &\int_{t_{j-1}}^{t_j} |\hat{\mathcal{N}}\big(\eta_\varepsilon(t),u_\varepsilon(t),y(t),v(t)\big) - \hat{\mathcal{N}}\big(⨍_j\eta, ⨍_j u, y(t), v(t)\big)|\,\mathrm{d}t \\ &\le C_{\hat{\mathcal{N}}}\Big(\sup_{t\in[t_{j-1},t_j]} \omega(|u_\varepsilon(t) - ⨍_j u|)(1+\|v\|^p_{\mathrm{L}^p(t_{j-1},t_j)} + \|\eta_\varepsilon\|^{p'}_{\mathrm{L}^{p'}(t_{j-1},t_j)} + \|\eta\|^{p'}_{\mathrm{L}^p(t_{j-1},t_j)}) \\ &\qquad + (|t_j - t_{j-1}|^{1/p} + \|\eta_\varepsilon\|^{p'-1}_{\mathrm{L}^{p'}(t_{j-1},t_j)} + \|\eta\|^{p'-1}_{\mathrm{L}^{p'}(t_{j-1},t_j)})\|\eta_\varepsilon - ⨍_j\eta\|_{\mathrm{L}^{p'}(t_{j-1},t_j)}\Big). \end{aligned}$$

Here $y(t)$ is a placeholder for $\frac{1}{\varepsilon}u_\varepsilon(t)$ and $v=\dot{u}_\varepsilon$. Note that by uniform convergence of u_ε and strong convergence of η_ε we obtain

$$\int_0^T |\hat{\mathcal{N}}\big(\eta_\varepsilon(t),u_\varepsilon(t),y(t),v(t)\big) - \hat{\mathcal{N}}\big(⨍\eta, ⨍u, y(t),v(t)\big)|\,\mathrm{d}t \to 0$$

as $\varepsilon\searrow 0$ and $\sup|t_j - t_{j-1}|\searrow 0$. Here $⨍h$ is the piecewise constant function with $⨍h = ⨍_j h$ on (t_{j-1},t_j). Hence, it remains to show

$$\liminf_{\varepsilon\searrow 0}\int_0^T \hat{\mathcal{N}}\big(⨍\eta, ⨍u, y(t), v(t)\big)\,\mathrm{d}t \ge \mathfrak{J}_0(u,\eta)\,\mathrm{d}t \quad \text{as } \varepsilon\searrow 0 \text{ and } \sup|t_j - t_{j-1}|\searrow 0.$$

By [Bra02, Thm 3.1] we obtain that

$$\liminf_{\varepsilon\searrow 0}\int_{t_{j-1}}^{t_j}\hat{\mathcal{N}}\big(⨍_j\eta, ⨍_j u, y(t), v(t)\big)\,\mathrm{d}t \geq \int_{t_{j-1}}^{t_j} G_{\mathrm{eff}}^{(j)}(\dot{u})\,\mathrm{d}t$$

with $G_{\mathrm{eff}}^{(j)}$ obtained from $g^{(j)}(y,v) = \hat{\mathcal{N}}\big(⨍_j\eta, ⨍_j u, y(t), v(t)\big)$. Applying Corollary 3.5 we obtain

$$\sum_j \int_{t_{j-1}}^{t_j}\big|\mathcal{M}\big(u(t),\dot{u}(t),\eta(t)\big) - G_{\mathrm{eff}}^{(j)}\big(\dot{u}(t)\big)\big|\,\mathrm{d}t \to 0$$

as $\sup|t_j - t_{j-1}| \searrow 0$. □

The Γ-limsup estimate is proven on a dense subset. We take piecewise affine functions as an approximation of $\widehat{u}$ and piecewise constant functions as an approximation of $\widehat{\eta}$. However, the oscillating ε scale is taken into account via the shape functions which solve the cell problem (3.12).

Proposition 3.7 (Recovery sequence)**.** *For all pairs* $(\widehat{u},\widehat{\eta}) \in \mathrm{W}^{1,p}(0,T)\times \mathrm{L}^{p'}(0,T)$ *there exists a recovery sequence* $\widehat{u}_\varepsilon \rightharpoonup \widehat{u}$ *in* $\mathrm{W}^{1,p}(0,T)$ *such that for all* $\widehat{\eta}_\varepsilon \to \widehat{\eta}$ *in* $\mathrm{L}^{p'}(0,T)$ *we have* $\mathfrak{J}_\varepsilon(\widehat{u}_\varepsilon,\widehat{\eta}_\varepsilon) \to \mathfrak{J}_0(\widehat{u},\widehat{\eta})$.

Proof. Step 1: Continuity of $\mathfrak{J}_0$. By convexity and p-growth of $v \mapsto \mathcal{M}(u,v,\eta)$ (see Lemma 3.1) and the continuity of $\mathcal{M}$ established in Corollary 3.5, we obtain that $\mathfrak{J}_0 : \mathrm{W}^{1,p}(0,T)\times \mathrm{L}^{p'}(0,T) \to \mathbb{R}$ is continuous in the norm topology.

Thus, by standard arguments of Γ-convergence (see e.g. [Bra02, Remark 1.29, Prop. 1.44], Lemma 1.2) it suffices to provide the construction of a recovery sequences for $(\widehat{u},\widehat{\eta})$ in a subset of $\mathrm{W}^{1,p}(0,T)\times \mathrm{L}^{p'}(0,T)$ that is dense in the norm topology.

Step 2: Restriction to a dense subset $D \subset \mathrm{W}^{1,p}(0,T)\times \mathrm{L}^{p'}(0,T)$. We define D as follows. We consider dyadic partitions $\{\, t_{j,N} := kT/2^N \mid k = 0,...,2^N \,\}$ of $[0,T]$ and assume that pairs $(\widehat{u},\widehat{\eta})$ in D are such that $\dot{\widehat{u}}$ and $\widehat{\eta}$ are constant on the intervals $]t_{j-1,N}, t_{j,N}[$. Moreover, we assume that the slopes $v_{j,N} = \dot{\widehat{u}}(t)$ for $t \in\,]t_{j-1,N}, t_{j,N}[$ are non-zero. By standard arguments we see that D is dense in $\mathrm{W}^{1,p}(0,T)\times \mathrm{L}^{p'}(0,T)$.

As all $\mathfrak{J}_\varepsilon$ and $\mathfrak{J}_0$ are integral functionals it is now sufficient to give the recovery construction of a $(\widehat{u},\widehat{\eta}) \in D$ on one subinterval $[t_{j-1,N}, t_{j,N}]$. For $\widehat{u}$ we take care that the values at both ends remain unchanged, so that joining the different constructions stays in $\mathrm{W}^{1,p}(0,T)$. Let $z_{u,v,\eta}$ solve the scaled minimization problem

$$\mathcal{M}(u,v,\eta) = |v|\int_0^{1/|v|}\hat{\mathcal{N}}\big(\eta, u, z_{u,v,\eta}(s), \dot{z}_{u,v,\eta}(s)\big)\,\mathrm{d}s \tag{3.13}$$

where $\hat{\mathcal{N}}$ is given in Lemma 3.2 and $z_{u,v,\eta} \in \mathrm{W}^{1,p}_v(0,1/|v|)$ for $v \neq 0$. Without loss of generality we assume $z_{u,v,\eta}(0) = 0$.

Step 3: Recovery construction. To simplify notation we neglect the dependence on j and N and write $[a,b] = [t_{j-1,N}, t_{j,N}]$, $v := \frac{1}{b-a}(\widehat{u}(b) - \widehat{u}(a))$. We use the shape functions $z_k^\varepsilon(t,s) = z_{u_k^\varepsilon,v,\eta_k^\varepsilon}(s)$ introduced in (3.13) where $t \in [a_{k-1}^\varepsilon, a_k^\varepsilon]$, $\eta_k^\varepsilon = \fint_{a_{k-1}^\varepsilon}^{a_k^\varepsilon} \widehat{\eta}\,\mathrm{d}t$ and $u_k^\varepsilon = \fint_{a_{k-1}^\varepsilon}^{a_k^\varepsilon} \widehat{u}\,\mathrm{d}t$. The points $(a_k^\varepsilon)_{k\in\mathbb{N}_0}$ introduce an intermediate scale $\varepsilon^{1/2}$ via $a_k^\varepsilon := a + k(b-a)/n_\varepsilon$ where

$$n_\varepsilon := \Big\lfloor \frac{b-a}{\varepsilon^{1/2}} \Big\rfloor \quad \text{(floor function)}.$$

Abbreviating $\hat{u}_k = \widehat{u}(a_k^\varepsilon)$ for $k = 0, 1, ..., n_\varepsilon$ we define the approximation $\widehat{u}_\varepsilon : [a_{k-1}^\varepsilon, a_k^\varepsilon] \to \mathbb{R}$ via

$$\widehat{u}_\varepsilon(t) = \varepsilon z_{u_k^\varepsilon,v,\eta_k^\varepsilon}\big(\tfrac{1}{\varepsilon}(t - a_{k-1}^\varepsilon)\big) + \varepsilon\Big\lfloor \frac{\hat{u}_{k-1}^\varepsilon}{\varepsilon} \Big\rfloor \quad \text{for} \quad a_{k-1}^\varepsilon \le t \le x_k^\varepsilon.$$

where $x_k^\varepsilon := a_{k-1}^\varepsilon + \frac{\varepsilon}{|v|}\big\lfloor \frac{|v|(a_k^\varepsilon - a_{k-1}^\varepsilon - \varepsilon^{3/4})}{\varepsilon} \big\rfloor$. On the remaining interval $[x_k^\varepsilon, a_k^\varepsilon]$ we define $\widehat{u}_\varepsilon$ to be the affine interpolation with

$$\widehat{u}_\varepsilon(x_k^\varepsilon) = v(x_k^\varepsilon - a_{k-1}^\varepsilon) + \varepsilon\Big\lfloor \frac{\hat{u}_{k-1}^\varepsilon}{\varepsilon} \Big\rfloor \quad \text{and} \quad \widehat{u}_\varepsilon(a_k^\varepsilon) = \varepsilon\Big\lfloor \frac{\hat{u}_k^\varepsilon}{\varepsilon} \Big\rfloor.$$

By construction we obtain

$$\int_{a_{k-1}^\varepsilon}^{x_k^\varepsilon} \hat{\mathcal{N}}(\eta_k^\varepsilon, u_k^\varepsilon, \varepsilon^{-1}\widehat{u}_\varepsilon, \dot{\widehat{u}}_\varepsilon)\,\mathrm{d}t = (x_k^\varepsilon - a_{k-1}^\varepsilon)\mathcal{M}(u_k^\varepsilon, v, \eta_k^\varepsilon)$$

Using that $\widehat{u}$ is affine on $[a,b]$ we obtain

$$\varepsilon^{3/4} \le a_k^\varepsilon - x_k^\varepsilon \le \frac{\varepsilon}{|v|} + \varepsilon^{3/4} \quad \text{and} \quad |\widehat{u}_\varepsilon(x_k^\varepsilon) - \widehat{u}_\varepsilon(x_{k-1}^\varepsilon)| \le 3\varepsilon + |v|\varepsilon^{3/4}. \tag{3.14}$$

Hence, $|\dot{\widehat{u}}_\varepsilon| \le 3\varepsilon^{1/4} + |v|$ on the interval $[x_k^\varepsilon, a_k^\varepsilon]$. Thus, using that $\|\eta_k^\varepsilon\|_{\mathrm{L}^{p'}}$ is bounded we obtain

$$\sum_{k=1}^{n_\varepsilon} \left((a_k^\varepsilon - x_k^\varepsilon)(\mathcal{M}(u_k^\varepsilon, v, \eta_k^\varepsilon) + \int_{x_k^\varepsilon}^{a_k^\varepsilon} \hat{\mathcal{N}}(\eta_k^\varepsilon, u_k^\varepsilon, \varepsilon^{-1}\widehat{u}_\varepsilon, \dot{\widehat{u}}_\varepsilon)\,\mathrm{d}t \right) \to 0.$$

Hence

$$\lim_\varepsilon \int_a^b \hat{\mathcal{N}}(\fint_\varepsilon \eta, \fint_\varepsilon u, \varepsilon^{-1}\widehat{u}_\varepsilon, \dot{\widehat{u}}_\varepsilon)\,\mathrm{d}t = \lim_\varepsilon \int_a^b \mathcal{M}(\fint_\varepsilon u, v, \fint_\varepsilon \eta)\,\mathrm{d}t$$

where $(\fint_\varepsilon h)(t) = \fint_{a_{k-1}^\varepsilon}^{a_k^\varepsilon} \widehat{h}\,\mathrm{d}t$ for $t \in [a_{k-1}^\varepsilon, a_k^\varepsilon]$. Corollary 3.5 finally gives

$$\lim_\varepsilon \int_a^b \hat{\mathcal{N}}(\fint_\varepsilon \eta, \fint_\varepsilon u, \varepsilon^{-1}\widehat{u}_\varepsilon, \dot{\widehat{u}}_\varepsilon)\,\mathrm{d}t = \int_a^b \mathcal{M}(\widehat{u}, v, \widehat{\eta})\,\mathrm{d}t.$$

By Lemma 3.2 we justify the modification $\widehat{u}_\varepsilon(t) \mapsto u_k^\varepsilon$ and $\widehat{\eta}_\varepsilon \mapsto \eta_k^\varepsilon$ by uniform convergence and strong $\mathrm{L}^{p'}$-convergence respectively. Indeed, we have $\widehat{u}_\varepsilon \rightharpoonup \widehat{u}$ in $\mathrm{W}^{1,p}(0,T)$. This can be seen as follows.

Because of the monotonicity of z_{u_k,v,η_k} and $z_{u_k,v,\eta_k}(m/V) = m$ for $m \in \mathbb{Z}$ we have the obvious estimate $|z_{u_k,v,\eta_k}(s) - vs| \leq 1$ which implies $|\widehat{u}_\varepsilon(t) - \widehat{u}(t)| \leq \varepsilon$ for $t \in \cup_k [a_{k-1}^\varepsilon, x_k^\varepsilon]$. Since both, $\widehat{u}$ and $\widehat{u}_\varepsilon$ are affine on $[x_k^\varepsilon, a_k^\varepsilon]$ and

$$|\widehat{u}(x_k^\varepsilon) - \widehat{u}_\varepsilon(x_k^\varepsilon)| = |\hat{u}_{k-1} - \lfloor \frac{\hat{u}_{k-1}^\varepsilon}{\varepsilon} \rfloor| \leq \varepsilon, \quad |\widehat{u}(a_k^\varepsilon) - \widehat{u}_\varepsilon(a_k^\varepsilon)| = |\hat{u}_k - \lfloor \frac{\hat{u}_k^\varepsilon}{\varepsilon} \rfloor| \leq \varepsilon.$$

Hence, $\|\widehat{u}_\varepsilon - \widehat{u}\|_{\mathrm{L}^\infty(a,b)} \leq \varepsilon$. As shown in Step 6 of the proof of Proposition 3.4 we have that

$$\int_{a_{k-1}^\varepsilon}^{x_k^\varepsilon} |\dot{\widehat{u}}_\varepsilon|^p \,\mathrm{d}t = \frac{\varepsilon k_\varepsilon}{|v|} \fint_0^{k_\varepsilon/|v|} |\dot{z}_{u_k,v,\eta_k}|^p \,\mathrm{d}s \leq \frac{\varepsilon k_\varepsilon}{|v|} c\big(1 + |\eta_k|^{p'} + |v|^p\big).$$

Here $k_\varepsilon = \frac{|v|}{\varepsilon}(x_k^\varepsilon - a_{k-1}^\varepsilon)$. Hence,

$$\int_a^b |\dot{\widehat{u}}_\varepsilon|^p \,\mathrm{d}t \leq c\|\eta\|_{\mathrm{L}^{p'}} + c(b-a)(1 + |v|^p) + (b-a)(\varepsilon^{1/2}/|v| + \varepsilon^{1/4})(3\varepsilon^{1/4} + |v|),$$

where we used the estimates (3.14) on the remaining intervals $\cup_k [x_k^\varepsilon, a_k^\varepsilon]$ and $n_\varepsilon \leq (b-a)\varepsilon^{-1/2}$. Thus we have $\widehat{u}_\varepsilon \rightharpoonup \widehat{u}$ in $\mathrm{W}^{1,p}(0,T)$. □

In particular, for fixed $\xi \in \mathbb{R}$ we obtain the following Γ-convergence result for the tilted total dissipation functional.

Corollary 3.8. *We have Γ-convergence of $\mathfrak{D}_\varepsilon^\zeta$ with respect to the weak $\mathrm{W}^{1,p}(0,T)$ topology, with $\mathfrak{D}_\varepsilon^\zeta$ defined in* (1.8)*, to*

$$\mathfrak{D}_0^\xi : u \mapsto \int_0^T \mathcal{M}(u, \dot{u}, -\mathrm{D}\mathcal{E}_0(t,u) + \xi) \,\mathrm{d}t.$$

3.2 Properties of the contact potential $\mathcal{M}$

By the established Γ-liminf estimate we arrive at

$$\mathcal{E}_0\big(T, u(T)\big) + \int_0^T \mathcal{M}\big(u(t), \dot{u}(t), -\mathrm{D}\mathcal{E}_0(t,u)\big) \,\mathrm{d}t \leq \mathcal{E}_0\big(0, u(0)\big). \tag{3.15}$$

Yet, it is not obvious what flow is induced by (3.15). Therefore, in this section we discuss the properties of $\mathcal{M}$. Most importantly we show the estimate $\mathcal{M}(u,v,\xi) \geq \xi v$ and characterize its contact set $\mathrm{C}_\mathcal{M}$ defined in (1.9). Moreover, we drop the dependence on the variable u, as it is simply playing the role of a fixed parameter. Hence, we abbreviate $\mathfrak{p}(y) = \partial_y \kappa(u,y)$. Note that $\mathfrak{p}$ is a continuous 1-periodic function with $\int_0^1 \mathfrak{p} \,\mathrm{d}y = 0$ with its extreme values denoted by

$$\overline{\mathfrak{p}} := \max\{\, \mathfrak{p}(y) \mid y \in \mathbb{R} \,\} \quad \text{and} \quad \underline{\mathfrak{p}} := \min\{\, \mathfrak{p}(y) \mid y \in \mathbb{R} \,\}.$$

The definition of $\mathcal{M}$ with the new notation reads

$$\mathcal{M}(v,\eta) = \inf_{z\in \mathrm{W}^{1,p}_v} \Big(\int_0^1 \big[\mathcal{R}(|v|\dot z(s)) + \mathcal{R}^*\big(\eta - \mathfrak{p}\big(z(s)\big)\big)\big]\,\mathrm{d}s\Big),$$

where $\mathrm{W}^{1,p}_v = \big\{v \in \mathrm{W}^{1,p}(0,1)\big| z(1) = z(0)+\mathrm{sign}(v)\big\}$. As a result $\mathcal{M}$ is defined in terms of $\mathcal{R}+\mathcal{R}^*$. We derive the following basic properties.

Lemma 3.9 (Basic properties of $\mathcal{M}$)**.** *(a) For all v,ξ we have $\mathcal{M}(v,\xi) \geq v\xi$.*
(b) For all $\xi \in \mathbb{R}$ we have

$$\mathcal{M}(0,\xi) = \min_{\pi\in[\underline{\mathfrak{p}},\overline{\mathfrak{p}}]} \mathcal{R}^*(\xi-\pi) \quad \textit{and} \quad \mathcal{M}(v,\xi) \geq \mathcal{M}(0,\xi) \textit{ for all } v.$$

(c) If $\mathcal{R}(-v) = \mathcal{R}(v)$ for all v, then also $\mathcal{M}(-v,\xi) = \mathcal{M}(v,\xi)$ for all $v,\xi \in \mathbb{R}$. If additionally, $\mathfrak{p}(y) = -\mathfrak{p}(y_*-y)$ for some y_* and all y, then also $\mathcal{M}(v,-\xi) = \mathcal{M}(v,\xi)$.

Proof. Part (a). For a minimizer z for $\mathcal{M}(v,\xi)$, we simply apply the Young-Fenchel inequality to the integrand in the definition of $\mathcal{M}$ and use that $\mathfrak{p}$ has mean 0:

$$\begin{aligned}\mathcal{M}(v,\xi) = \int_0^1 \big(\mathcal{R}(|v|\dot z)+\mathcal{R}^*(\xi-\mathfrak{p}(z))\big)\,\mathrm{d}s \\ \geq \int_0^1 |v|\dot z(s)(\xi-\mathfrak{p}(z(s)))\big)\,\mathrm{d}s = |v|(z(1)-z(0))\xi.\end{aligned}$$

Because of $z(1) = z(0) + \mathrm{sign}(v)$ we obtain the desired result.

Part (b). The result for $v = 0$ is trivial, as we can choose a constant minimizer $z(s) = z_*$. When comparing $v = 0$ and $v \neq 0$ we take a minimizer for $z_{v,\xi}$ and estimate

$$\mathcal{M}(v,\xi) = \int_0^1 \big(\mathcal{R}(|v|\dot z_{v,\xi})+\mathcal{R}^*(\xi-\mathfrak{p}(z_{v,\xi}))\big)\,\mathrm{d}s \geq \int_0^1 \min_{\pi\in[\underline{\mathfrak{p}},\overline{\mathfrak{p}}]} \mathcal{R}^*(\xi-\pi)\,\mathrm{d}s = \mathcal{M}(0,\xi).$$

Part (c). The first symmetry follows since minimizers $z_{v,\xi}$ give minimizers $z_{-v,\xi} : s \mapsto z_{v,\xi}(1-s)$ and vice versa. For the second symmetry we consider $z_{v,-\xi} : s \mapsto y_* - z_{v,\xi}(s)$. ☐

The next result is concerned with the contact set $\mathbf{C}_\mathcal{M} = \{(v,\xi)|\mathcal{M}(v,\xi) = \xi v\}$. As mentioned in the explanations of Section 1.3 we know that the limit evolution is a subset of $\mathbf{C}_\mathcal{M}$, i.e. for almost all $t \in (0,T)$ we have

$$\big(\dot u(t), -\mathrm{D}\mathcal{E}_0(t,u(t))\big) \in \mathbf{C}_\mathcal{M}.$$

Moreover, we give an explicit kinetic relation

$$(v,\xi) \in \mathcal{M} \Longleftrightarrow v = \mathrm{K}(\xi).$$

Since $\mathrm{K} : \mathbb{R} \mapsto \mathbb{R}$ we have a primitive $\mathcal{R}^*_{\text{eff}}$ of $\mathrm{K} = \partial\mathcal{R}^*_{\text{eff}}$. In particular, the flow is induced by the gradient system $(\mathbb{R}, \mathcal{E}_0, \mathcal{R}_{\text{eff}})$.

Proposition 3.10 (Effective dissipation potential). *There is a unique effective dissipation potential* $\mathcal{R}_{\text{eff}} : \mathbb{R} \to \mathbb{R}$ *such that*

$$\mathsf{C}_{\mathcal{M}} = \text{graph}(\partial\mathcal{R}_{\text{eff}}) = \big\{(v,\xi) | \xi \in \partial\mathcal{R}_{\text{eff}}(v)\big\} = \big\{(v,\xi) | \mathcal{R}_{\text{eff}}(v) + \mathcal{R}^*_{\text{eff}}(\xi) = v\xi\big\}. \quad (3.16)$$

If $\mathcal{R}$ *is strictly convex (and hence,* $\mathcal{R}^*$ *differentiable), then the potential* $\mathcal{R}_{\text{eff}}$ *is characterized by the fact that* $\partial\mathcal{R}^*_{\text{eff}}(\xi)$ *is the harmonic mean of the functions* $[0,1] \in y \mapsto \partial\mathcal{R}^*(\xi-\mathfrak{p}(y))$, *viz.*

$$\partial\mathcal{R}^*_{\text{eff}}(\xi) = \begin{cases} \{0\} & \textit{for } \underline{\mathfrak{p}} < \xi < \overline{\mathfrak{p}}, \\ \{\mathrm{K}(\xi)\} & \textit{for } \xi < \underline{\mathfrak{p}} \textit{ or } \xi > \overline{\mathfrak{p}}, \\ [0, \mathrm{K}(\overline{\mathfrak{p}})] & \textit{for } \xi = \overline{\mathfrak{p}}, \\ [\mathrm{K}(\underline{\mathfrak{p}}), 0] & \textit{for } \xi = \underline{\mathfrak{p}}, \end{cases} \quad \textit{where } \mathrm{K}(\xi) = \left(\int_0^1 \frac{\mathrm{d}y}{\partial\mathcal{R}^*(\xi-\mathfrak{p}(y))}\right)^{-1}.$$

Proof. As shown in the proof of Lemma 3.9(a), $\mathcal{M}(v,\xi) = \xi v$ can only hold if the minimizer $z_{v,\xi}$ satisfies

$$\mathcal{R}(|v|\dot{z}_{v,\xi}(s)) + \mathcal{R}^*(\xi - \mathfrak{p}(z_{v,\xi}(s))) = |v|\dot{z}_{v,\xi}(s)\,\big(\xi-\mathfrak{p}(z_{v,\xi}(s))\big) \text{ for a.a. } s \in [0,1].$$

By the Fenchel equivalences $z = z_{v,\xi}$ has to satisfy the differential inclusion

$$|v|\dot{z}(s) \in \partial\mathcal{R}^*\big(\xi - \mathfrak{p}(z(s))\big), \quad z(1) = z(0) + \text{sign}(v). \quad (3.17)$$

In particular,

$$\underline{\mathfrak{p}} < \xi < \overline{\mathfrak{p}} \Longrightarrow v = 0.$$

Since by monotonicity of z the left hand side has a fixed sign, whereas the right hand side has a changing sign due to the boundary conditions of z. For $\xi < \underline{\mathfrak{p}}$ or $\xi > \overline{\mathfrak{p}}$ we have $\partial\mathcal{R}^*\big(\xi - \mathfrak{p}(z(s))\big) \neq 0$. Thus we solve the equation via separation of the variables z and s, and the boundary condition gives

$$1 = \int_0^1 \mathrm{d}s = \int_0^1 \frac{|v|\dot{z}(s)\,\mathrm{d}s}{\partial\mathcal{R}^*\big(\xi-\mathfrak{p}(z(s))\big)} = |v|\text{sign}(v)\int_0^1 \frac{\mathrm{d}y}{\partial\mathcal{R}^*\big(\xi-\mathfrak{p}(y)\big)} = \frac{v}{\mathrm{K}(\xi)}.$$

Here we used the continuity of $\partial\mathcal{R}^*$. In particular,

$$\xi < \underline{\mathfrak{p}} \leq \overline{\mathfrak{p}} < \xi \Longrightarrow v = \mathrm{K}(\xi).$$

We observe that $\xi \mapsto K(\xi)$ is monotone and $\xi K(\xi) \geq 0$. Hence, $\mathcal{R}^*_{\text{eff}}(\xi) = \int_0^\xi K(\eta)\,\mathrm{d}\eta$ gives the desired dual effective dissipation potential. Defining $\mathcal{R}_{\text{eff}}$ by Legendre transform, the Fenchel equivalences provide the desired relation between $\mathsf{C}_{\mathcal{M}}$ and the graph of $\mathcal{R}_{\text{eff}}$ except for $\xi \in \{\underline{\mathfrak{p}}, \overline{\mathfrak{p}}\}$. We prove

$$\xi = \overline{\mathfrak{p}} \Longrightarrow v \in [0, \mathrm{K}(\overline{\mathfrak{p}})]$$

only, since the case $\xi = \underline{\mathfrak{p}}$ is analogous. By continuity we easily obtain

$$(\overline{\mathfrak{p}} + \delta)\mathrm{K}(\overline{\mathfrak{p}} + \delta) = \mathcal{M}(\mathrm{K}(\overline{\mathfrak{p}} + \delta), \overline{\mathfrak{p}} + \delta) \longrightarrow \overline{\mathfrak{p}}\mathrm{K}(\overline{\mathfrak{p}}) = \mathcal{M}(\mathrm{K}(\overline{\mathfrak{p}}), \overline{\mathfrak{p}}) \quad \text{as } \delta \searrow 0,$$

i.e, $(\mathrm{K}(\overline{\mathfrak{p}}),\overline{\mathfrak{p}}) \in \mathbf{C}_{\mathcal{M}}$. Let $\sigma_h = h\mathrm{K}(\overline{\mathfrak{p}})$ with $0 < h < 1$ We take the minimizer $z_{\mathrm{K}(\overline{\mathfrak{p}}),\overline{\mathfrak{p}}}$ and let t_* such that $\mathfrak{p}\big(z_{\mathrm{K}(\overline{\mathfrak{p}}),\overline{\mathfrak{p}}}(t_*)\big) = \overline{\mathfrak{p}}$. We obtain a minimizer $z_{\sigma_h,\overline{\mathfrak{p}}}$ by

$$z_{\sigma_h,\overline{\mathfrak{p}}} = \begin{cases} z_{\mathrm{K}(\overline{\mathfrak{p}}),\overline{\mathfrak{p}}}(t/h) & \text{for } 0 \le t < ht_*, \\ z_{\mathrm{K}(\overline{\mathfrak{p}}),\overline{\mathfrak{p}}}(t_*) & \text{for } ht_* \le t \le 1 + h(t_* - 1), \\ z_{\mathrm{K}(\overline{\mathfrak{p}}),\overline{\mathfrak{p}}}(t/h + 1 - h) & \text{for } 1 + h(t_* - 1) < t \le 1 \end{cases}$$

We observe that

$$\int_{ht_*}^{1+h(t_*-1)} \mathcal{R}(|\sigma_h|\dot{z}_{\sigma_h,\overline{\mathfrak{p}}}) + \mathcal{R}^*\big(\overline{\mathfrak{p}} - \mathfrak{p}(z_{\sigma_h,\overline{\mathfrak{p}}})\big)\,\mathrm{d}t = 0 \tag{3.18a}$$

and for $t < ht_*$ and $1 + h(t_* - 1) < t$ we have $h\dot{z}_{\sigma_h,\overline{\mathfrak{p}}}(t) = \dot{z}_{\mathrm{K}(\overline{\mathfrak{p}}),\overline{\mathfrak{p}}}(t/h)$. With the latter identity we compute

$$\begin{aligned} &\int_0^{t_*} \mathcal{R}(|\mathrm{K}(\overline{\mathfrak{p}})|\dot{z}_{\mathrm{K}(\overline{\mathfrak{p}}),\overline{\mathfrak{p}}}(t)) + \mathcal{R}^*\big(\overline{\mathfrak{p}} - \mathfrak{p}(z_{\mathrm{K}(\overline{\mathfrak{p}}),\overline{\mathfrak{p}}}(t))\big)\,\mathrm{d}t \\ &= \frac{1}{h}\int_0^{ht_*} \mathcal{R}(|\mathrm{K}(\overline{\mathfrak{p}})|\dot{z}_{\mathrm{K}(\overline{\mathfrak{p}}),\overline{\mathfrak{p}}}(\tau/h)) + \mathcal{R}^*\big(\overline{\mathfrak{p}} - \mathfrak{p}(z_{\mathrm{K}(\overline{\mathfrak{p}}),\overline{\mathfrak{p}}}(\tau/h))\big)\,\mathrm{d}\tau \\ &= \frac{1}{h}\int_0^{ht_*} \mathcal{R}(|\sigma_h|\dot{z}_{\sigma_h,\overline{\mathfrak{p}}}(t)) + \mathcal{R}^*\big(\overline{\mathfrak{p}} - \mathfrak{p}(z_{\sigma_h,\overline{\mathfrak{p}}}(t))\big)\,\mathrm{d}t. \end{aligned} \tag{3.18b}$$

Similarly, we obtain

$$\begin{aligned} &\int_{t_*}^{1} \mathcal{R}(|\mathrm{K}(\overline{\mathfrak{p}})|\dot{z}_{\mathrm{K}(\overline{\mathfrak{p}}),\overline{\mathfrak{p}}}(t)) + \mathcal{R}^*\big(\overline{\mathfrak{p}} - \mathfrak{p}(z_{\mathrm{K}(\overline{\mathfrak{p}}),\overline{\mathfrak{p}}}(t))\big)\,\mathrm{d}t \\ &= \frac{1}{h}\int\limits_{1+h(t_*-1)}^{1} \mathcal{R}(|\sigma_h|\dot{z}_{\sigma_h,\overline{\mathfrak{p}}}(t)) + \mathcal{R}^*\big(\overline{\mathfrak{p}} - \mathfrak{p}(z_{\sigma_h,\overline{\mathfrak{p}}}(t))\big)\,\mathrm{d}t. \end{aligned} \tag{3.18c}$$

Combining all three identities (3.18) we obtain

$$\begin{aligned} \overline{\mathfrak{p}}\mathrm{K}(\overline{\mathfrak{p}}) &= \int_0^1 \mathcal{R}(|\mathrm{K}(\overline{\mathfrak{p}})|\dot{z}_{\mathrm{K}(\overline{\mathfrak{p}}),\overline{\mathfrak{p}}}(t)) + \mathcal{R}^*\big(\overline{\mathfrak{p}} - \mathfrak{p}(z_{\mathrm{K}(\overline{\mathfrak{p}}),\overline{\mathfrak{p}}}(t))\big)\,\mathrm{d}t \\ &= \frac{1}{h}\int_0^1 \mathcal{R}(|\sigma_h|\dot{z}_{\sigma_h,\overline{\mathfrak{p}}}(t)) + \mathcal{R}^*\big(\overline{\mathfrak{p}} - \mathfrak{p}(z_{\sigma_h,\overline{\mathfrak{p}}}(t))\big)\,\mathrm{d}t \end{aligned}$$

This is exactly $\mathcal{M}(\sigma_h,\overline{\mathfrak{p}}) = \overline{\mathfrak{p}}\sigma_h$. □

Although the contact set is given by the subdifferential of $\mathcal{R}_{\mathrm{eff}}$ we have that $\mathcal{M}(\cdot_v,\cdot_\xi) \neq \mathcal{R}_{\mathrm{eff}}(\cdot_v) + \mathcal{R}^*_{\mathrm{eff}}(\cdot_\xi)$. For $v = 0$ and $0 < \overline{\mathfrak{p}} < \xi$ we obtain

$$\begin{aligned} \mathcal{R}^*_{\mathrm{eff}}(\xi) &= \int_{\overline{\mathfrak{p}}}^{\xi} \left(\int_0^1 \frac{1}{\partial\mathcal{R}^*\big(\eta - \mathfrak{p}(z)\big)}\,\mathrm{d}z\right)^{-1} \mathrm{d}\eta \\ &> \int_{\overline{\mathfrak{p}}}^{\xi} \left(\int_0^1 \frac{1}{\partial\mathcal{R}^*\big(\eta - \overline{\mathfrak{p}}\big)}\,\mathrm{d}z\right)^{-1} \mathrm{d}\eta = \mathcal{R}^*(\xi - \overline{\mathfrak{p}}) = \mathcal{M}(0,\xi). \end{aligned}$$

By Proposition 3.10 the limit evolution can be interpreted as a gradient flow induced by the gradient system $(\mathbb{R}, \mathcal{E}_0, \mathcal{R}_{\text{eff}})$. In particular, we obtain the following relaxed EDP-convergence result.

Theorem 3.11. *We have relaxed EDP-convergence of the gradient system* $(\mathbb{R}, \mathcal{E}_\varepsilon, \mathcal{R})$ *given by* (3.3) *to the effective gradient system* $(\mathbb{R}, \mathcal{E}_0, \mathcal{R}_{\text{eff}})$ *with* $\mathcal{R}_{\text{eff}}$ *given by* (3.16)

We now want to study the behavior of $\mathcal{M}(v, \xi)$ for small v, which emphasizes the sticking phenomenon induced by the wiggly energy landscape. To simplify the argument we assume that $\mathcal{R}$ behave like a power near $v = 0$, i.e., $\mathcal{R}^*(\xi) = 0 \Leftrightarrow \xi = 0$.

The proof involves an argument of Modica-Mortola type (cf. [MM77] and [Bra02, Ch. 6]) as for small velocities the minimizers z for $\mathcal{M}$ are mostly near minimizers for $y \mapsto \mathcal{R}^*(\xi - \mathfrak{p}(y))$ but have a transition layer of width $|v|$ to make a jump of size 1.

Lemma 3.12 (Expansion of $\mathcal{M}$ for $v \approx 0$)**.** *For* $v > 0$ *we have*

$$\mathcal{M}(v, \xi) = \mathcal{M}(0, \xi) \ + \ v\, M_1(\xi) + o(v) \text{ for } v \searrow 0, \tag{3.19}$$

with $M_1(\xi) = \int_0^1 \Psi\big(\mathcal{R}^*(\xi - \mathfrak{p}(y)) - \mathcal{M}(0, \xi)\big)\,\mathrm{d}y$*, where* $\Psi : [0, \infty[\to [0, \infty[$ *is the inverse function of* $\mathcal{R}^* : [0, \infty[\to [0, \infty[$*.*

In particular, for $\xi \in [\underline{\mathfrak{p}}, \overline{\mathfrak{p}}]$ *we have* $\mathcal{M}(0, \xi) = 0$ *and if additionally* $\mathcal{R}$ *is symmetric, then* $M_1(\xi) = \int_0^1 |\xi - \mathfrak{p}(y)|\,\mathrm{d}y$*.*

Proof. We fix ξ and choose $y_* \in \operatorname{argmin} \mathcal{R}^*(\xi - \mathfrak{p}(\cdot))$. We rewrite $\mathcal{M}(v, \xi)$ in the form

$$\mathcal{M}(v, \xi) = \mathcal{M}(0, \xi) + v M_1(v, \xi) \text{ with } M_1(\xi, v) = \min_{z(1) = z(0) + 1} \int_0^1 \frac{1}{v}\big(\mathcal{R}(v\dot{z}) + G_\xi(z(y))\big)\,\mathrm{d}s,$$

where $G_\xi(z) = \mathcal{R}^*(\xi - \mathfrak{p}(z)) - \mathcal{R}^*(\xi - \mathfrak{p}(y_*)) \geq 0$.

Setting $s = v\tau$ and $w(\tau) = z(v\tau)$ we see that w has to minimize $\int_0^{1/v} \big(\mathcal{R}(w'(\tau) + G_\xi(w(\tau))\big)\,\mathrm{d}\tau$ under the constraint $w(1/v) = w(0) + 1$. Indeed, by periodicity of $\mathfrak{p}$ in y we may assume $w(0) = y_*$, so we are in the classical Modica-Mortola setting of phase transitions.

We define H_ξ via the relation $G_\xi(z) = \mathcal{R}^*(H_\xi(z))$, i.e., $H_\xi(z) = \Psi(G_\xi(z))$. Now, the methods in [Bra02, Ch. 6] give the convergence $M_1(v, \xi) \to M_1(0, \xi)$ with

$$M_1(0, \xi) = \min_{\substack{w(-\infty) = y_*, \\ w(\infty) = y_* + 1}} \int_{\tau \in \mathbb{R}} \big[\mathcal{R}(w'(\tau)) + \mathcal{R}^*\big(H_\xi(w(\tau))\big)\big]\,\mathrm{d}\tau = \int_{y_*}^{y_* + 1} H_\xi(z)\,\mathrm{d}z.$$

Because of the periodicity of $\mathfrak{p}$ this is the desired formula for M_1.

The last statement follows if we use $\mathcal{R}^*(-\xi) = \mathcal{R}^*(\xi)$ which gives $\Psi(\mathcal{R}^*(\eta)) = |\eta|$. ☐

We finally look at the rate-independent limit that was already studied in [Mie12]. The relevant time rescaling is obtained by

$$\text{replacing } \mathcal{R} \text{ by } \mathcal{R}_\delta : v \mapsto \frac{1}{\delta}\mathcal{R}(\delta v),$$

where δ is a positive parameter that tends to 0 in the rate-independent limit, cf. [EM06, MRS09].

This scaling obviously gives $\mathcal{R}^*_\delta(\xi) = \frac{1}{\delta}\mathcal{R}^*(\xi)$, so that the associated rescaled effective contact potential is $\mathcal{M}_\delta(v,\xi) = \frac{1}{\delta}\mathcal{M}(\delta v,\xi)$. We obtain indeed the same result as in [Mie12, Prop. 3.1], where a joint limit was taken (i.e. $\delta_\varepsilon \searrow 0$ with $\varepsilon \searrow 0$) while our result is a double limit, where first $\varepsilon \to 0$ and then $\delta \to 0$.

Since $\mathcal{M}_\delta$ has p-growth and is almost positively homogenous of degree 1 near $\delta \approx 0$ the limit evolution can be seen as a balanced-viscosity solution (see e.g. [MS18, RS17]) to the corresponding rate-independent limit.

Corollary 3.13 (Rate-independent limit). *Under the above assumptions and $\mathcal{R}(-v) = \mathcal{R}(v)$ we have*

$$\mathcal{M}_\delta(v,\xi) \xrightarrow{\delta\to 0} \mathcal{M}_{\mathrm{RI}}(v,\xi) = \begin{cases} |v|M_1(\xi) & \text{for } \xi \in [\underline{\mathfrak{p}},\overline{\mathfrak{p}}], \\ \infty & \text{for } \xi \notin [\underline{\mathfrak{p}},\overline{\mathfrak{p}}], \end{cases} \quad \text{with } M_1(\xi) = \int_0^1 |\xi - \mathfrak{p}(y)|\,\mathrm{d}y.$$

Proof. Case $\xi \notin [\underline{\mathfrak{p}},\overline{\mathfrak{p}}]$. We have $\mathcal{M}_\delta(v,\xi) \geq \mathcal{M}_\delta(0,\xi) = \frac{1}{\delta}M_0(\xi)$. Because of $M_0(\xi) > 0$ for this case we are done.

Case $\xi \in [\underline{\mathfrak{p}},\overline{\mathfrak{p}}]$. We now have $M_0(\xi) = 0$, and Lemma 3.12 gives the result. □

Finally we discuss the kinetic relation $v = \partial\mathcal{R}^*_{\mathrm{eff}}(\xi)$ for ξ slightly outside the sticking region $[\underline{\mathfrak{p}},\overline{\mathfrak{p}}]$ and for very large ξ. For simplicity we restrict to the quadratic case.

Lemma 3.14 (Expansion of kinetic relation). *Assume that the dissipation potential is given by $\mathcal{R}(v) = \frac{1}{2}v^2$ and let $\mathfrak{p}$ have a unique maximizer z_* such that $\mathfrak{p}(z) = \overline{\mathfrak{p}} - c_*|z-z_*|^\alpha + O(|z-z_*|^\gamma)$ with $c_* > 0$, $1 < \alpha < \infty$, and $\gamma > 2\alpha - 1$. Then,*

$$K(\xi) = c_*^{1/\alpha} S_\alpha^{-1} \max\{0, \xi - \overline{\mathfrak{p}}\}^{(\alpha-1)/\alpha} + o(|\xi - \overline{\mathfrak{p}}|^{(\alpha-1)/\alpha}) \quad \text{for} \quad \xi \to \overline{\mathfrak{p}}$$

with $S_\alpha = 2\sum_0^\infty (-1)^n \big(\frac{1}{\alpha n+1} + \frac{1}{\alpha(n+1)-1}\big)$. In the case $\alpha = 2$, we have $S_2 = \pi$ and $K(\xi) = \sqrt{c_*}\pi^{-1}\big(\max\{0,\xi-\overline{\mathfrak{p}}\}\big)^{1/2} + o(|\xi-\overline{\mathfrak{p}}|^{1/2})$.

For general $\mathfrak{p}$ we obtain $K(\xi) - \xi \to 0$ as $|\xi| \to \infty$

Proof. The computation is performed only for $z \in [z_*, 1]$ since we are able to conclude by symmetry. We recall the definition of K for the case $\mathcal{R}(v) = \frac{1}{2}v^2$

$$\mathrm{K}(\xi) = \left(\int_0^1 \frac{\mathrm{d}y}{\xi - \mathfrak{p}(y)} \right)^{-1}.$$

We define $h(z) = \mathfrak{p}(z+z_*) - \overline{\mathfrak{p}} + c_*|z|^\alpha$. With $\delta > 0$ fixed and set $\varepsilon := \xi - \overline{\mathfrak{p}}$ we observe

$$\int_{z^*}^{z^*+\delta} \frac{1}{\varepsilon + \overline{\mathfrak{p}} - \mathfrak{p}(z)}\,\mathrm{d}z = \int_0^\delta \frac{1}{\varepsilon + c_* z^\alpha}\,\mathrm{d}z + \int_0^\delta \frac{1}{\frac{(\varepsilon + c_* z^\alpha)^2}{h(z)} + \varepsilon + c_* z^\alpha}\,\mathrm{d}z.$$

We want to argue only for the leading order term. Since $\gamma > 2\alpha - 1$ we have

$$0 \le \int_0^\delta \frac{1}{\frac{(\varepsilon + c_* z^\alpha)^2}{h(z)} + \varepsilon + c_* z^\alpha}\,\mathrm{d}z \le \int_0^\delta \frac{h(z)}{(\varepsilon + c_* z^\alpha)^2}\,\mathrm{d}z \le \int_0^\delta c_*^{-1} h(z) z^{-2\alpha}\,\mathrm{d}z \to 0$$

as $\delta \searrow 0$. Let $\delta_\varepsilon = (\varepsilon/c_*)^{1/\alpha}$. For $h < x$ we use the geometric series

$$\frac{1}{x+h} = \sum_{n=0}^\infty (-1)^n \frac{h^n}{x^{n+1}}. \tag{3.20}$$

With this we compute

$$\begin{aligned}\int_0^{\delta_\varepsilon} \frac{1}{\varepsilon + c_* z^\alpha}\,\mathrm{d}z &\overset{(3.20)}{=} \int_0^{\delta_\varepsilon} \sum_{n=0}^\infty (-1)^n \varepsilon^{-(n+1)} (c_* z^\alpha)^n\,\mathrm{d}z \\ &= \sum_{n=0}^\infty (-1)^n \varepsilon^{-(n+1)} \frac{c_*^n}{\alpha n + 1} \delta_\varepsilon^{\alpha n + 1} = c_*^{-1/\alpha} \varepsilon^{\frac{1}{\alpha} - 1} \sum_{n=0}^\infty (-1)^n \frac{1}{\alpha n + 1}.\end{aligned}$$

For the remaining interval we obtain

$$\begin{aligned}&\int_{\delta_\varepsilon}^{\delta} \frac{1}{\varepsilon + c_* z^\alpha}\,\mathrm{d}z \overset{(3.20)}{=} \int_{\delta_\varepsilon}^{\delta} \sum_{n=0}^\infty (-1)^n \frac{\varepsilon^n}{(c_* z^\alpha)^{n+1}}\,\mathrm{d}z \\ &= \sum_{n=0}^\infty (-1)^n \frac{\varepsilon^n}{c_*^{n+1}} \frac{1}{\alpha(n+1) - 1} \left(\left(\frac{c_*}{\varepsilon}\right)^{n+1} \left(\frac{\varepsilon}{c_*}\right)^{\frac{1}{\alpha}} - \frac{\delta}{\delta^{\alpha(n+1)}} \right) \\ &= c_*^{-1/\alpha} \varepsilon^{\frac{1}{\alpha} - 1} \left(\sum_{n=0}^\infty (-1)^n \frac{1}{\alpha(n+1) - 1} \right) - \sum_{n=0}^\infty (-1)^n \frac{1}{\alpha(n+1) - 1} \left(\frac{\varepsilon}{c_* \delta^\alpha}\right)^n \frac{\delta^{1-\alpha}}{c_*}.\end{aligned}$$

We set $\delta = \varepsilon^{\frac{1-\gamma}{\alpha}}$ for some $0 < \gamma < 1$. In particular, $\delta \gg \delta_\varepsilon$, $\varepsilon \delta^{-\alpha} = \varepsilon^\gamma$ and $\delta^{1-\alpha} \ll \varepsilon^{\frac{1}{\alpha} - 1}$. Similarly, $\int_{z_* + \delta}^1 \frac{1}{\varepsilon + \overline{\mathfrak{p}}(u) - \mathfrak{p}(z)}\,\mathrm{d}z = o(\varepsilon^{\frac{1}{\alpha} - 1})$. This leads to

$$\mathrm{K}(\xi) = c_*^{1/\alpha} S_\alpha^{-1} (\max\{0, \xi - \overline{\mathfrak{p}}\})_+^{1 - \frac{1}{\alpha}} + o(|\xi - \overline{\mathfrak{p}}\}|^{1 - \frac{1}{\alpha}})$$

with S_α as given above. For general $\mathfrak{p}$ the limit $|\xi| \to \infty$ yields

$$\begin{aligned}\mathrm{K}(\xi) - \xi &= \xi \frac{1 - \int_0^1 \left(1 - \frac{\mathfrak{p}(z)}{\xi}\right)^{-1} \mathrm{d}z}{\int_0^1 \left(1 - \frac{\mathfrak{p}(z)}{\xi}\right)^{-1} \mathrm{d}z} \\ &= \frac{\int_0^1 \mathfrak{p}(z) \frac{\xi}{\mathfrak{p}(z)} \left(1 - \left(1 - \frac{\mathfrak{p}(z)}{\xi}\right)^{-1}\right) \mathrm{d}z}{\int_0^1 \left(1 - \frac{\mathfrak{p}(z)}{\xi}\right)^{-1} \mathrm{d}z} \longrightarrow \frac{\int_0^1 -\mathfrak{p}(z)\,\mathrm{d}z}{\int_0^1 1\,\mathrm{d}z} = 0.\end{aligned}$$

This is the desired result. □

Finally, we look at the case that the maximum of $\mathfrak{p}$ is approached linearly, i.e., the limiting case $\alpha = 1$ that is excluded in the previous lemma.

Lemma 3.15. *Assume $\mathcal{R}(v) = \frac{1}{2}v^2$ and let $\mathfrak{p}$ have a unique maximum such that $\mathfrak{p}(z) = \overline{\mathfrak{p}} - c_*|z{-}z_*| + O(|z{-}z_*|^\gamma)$ with $\gamma > 1$, then*

$$\mathrm{K}(\xi) = \frac{c_*}{2}\Big(\log\big((\xi - \overline{\mathfrak{p}})_+^{-1}\big)\Big)^{-1} + o\Big(\big(\log\big((\xi - \overline{\mathfrak{p}})_+^{-1}\big)\big)^{-1}\Big) \quad \textit{as} \quad \xi \searrow \overline{\mathfrak{p}}.$$

Proof. As in the proof of the previous lemma the computation is performed only on $[z_*, 1]$ since we are able to conclude by symmetry. We define $h(z) = \mathfrak{p}(z + z_*) - \overline{\mathfrak{p}} + c_*|z|$. With $\delta > 0$ fixed we observe

$$\int_{z^*}^{z^*+\delta} \frac{1}{\varepsilon + \overline{\mathfrak{p}} - \mathfrak{p}(z)}\,\mathrm{d}z = \int_0^\delta \frac{1}{\varepsilon + c_* z}\,\mathrm{d}z + \int_0^\delta \frac{1}{\frac{(\varepsilon + c_* z)^2}{h(z)} + \varepsilon + c_* z}\,\mathrm{d}z.$$

We want to argue only for the leading order term. We have

$$0 \le \int_0^\delta \frac{1}{\frac{(\varepsilon + c_* z)^2}{h(z)} + \varepsilon + c_* z}\,\mathrm{d}z \le \int_0^\delta \frac{h(z)}{(\varepsilon + c_* z)^2}\,\mathrm{d}z \le \int_0^\delta c_*^{-1} h(z) z^{-2}\,\mathrm{d}z \to 0$$

as $\delta \searrow 0$. For the remaining term we compute

$$\int_0^\delta \frac{1}{\varepsilon + c_* z}\,\mathrm{d}z = c_*^{-1}\big(\log(\tfrac{1}{\varepsilon}) + \log(\varepsilon + c_*\delta)\big).$$

We set $\delta = \eta_\varepsilon$ such that $\varepsilon = o(\eta_\varepsilon)$. Then we have $\int_{z_*+\eta_\varepsilon}^1 \frac{1}{\varepsilon + \overline{\mathfrak{p}} - \mathfrak{p}(z)}\,\mathrm{d}z = o\big(\log(\frac{1}{\varepsilon})\big)$. $\square$

The following remark shows that $\partial\mathcal{R}_{\mathrm{eff}}^*$ need not be continuous.

Remark 3.16. *For $\mathfrak{p}(z) = \overline{\mathfrak{p}} - c_*|z{-}z_*|^\alpha + O(|z{-}z_*|^\gamma)$ with $c_* > 0$ and $0 < \alpha < 1$ the integrand $z \mapsto (\xi{-}\mathfrak{p}(z))^{-1}$ remains integrable for $\xi \searrow \overline{\mathfrak{p}}$, so that $\partial\mathcal{R}_{\mathrm{eff}}^*(\xi) \to \sigma_* > 0$. Hence, $\mathcal{R}_{\mathrm{eff}}^*$ is Lipschitz continuous, but not differentiable, and $\partial\mathcal{R}_{\mathrm{eff}}^*$ is multi-valued, namely $\partial\mathcal{R}_{\mathrm{eff}}^*(\overline{\mathfrak{p}}) = [0, \sigma_*]$.*

For general power-law potentials $\mathcal{R}(v) = \frac{1}{p}|v|^p$ we transfer the results by observing

$$\big(|h|^\alpha + O(|h|^\gamma)\big)^{p-1} = |h|^{\alpha(p-1)} + O(|h|^{\alpha(p-2)-\gamma}).$$

Hence, we have the regimes $\alpha > p'{-}1$ with $\gamma > \alpha p{-}1$ and $\alpha = p'{-}1$ with $\gamma > p'{-}1$.

4 Diffusion in a thin sandwich-like domain

In this chapter we discuss two limit passages of a diffusion equation with drift in a thin domain. The equation

$$\dot{\mathbf{u}} = \mathrm{div}(\mathrm{A}_\varepsilon \nabla \mathbf{u}) \quad \text{in } \Omega_\varepsilon \quad \text{and} \quad \mathrm{A}_\varepsilon \nabla \mathbf{u} \cdot \nu_\varepsilon = 0 \quad \text{on } \partial\Omega_\varepsilon \quad \text{with} \quad \mathbf{u}(0) = \mathbf{u}_0 \tag{4.1}$$

can be formulated via two different gradient systems. We demonstrate that the effective dissipation potential may not be quadratic although the $\hat{\mathcal{R}}_\varepsilon^{(j)*}$ (see below) are quadratic. This depends on the choice of the gradient system. We consider two gradient systems. First, the H^{-1}-gradient system

$$\begin{aligned}
\mathbf{X}_\varepsilon^{(1)} &= \big(\mathrm{H}^1_{\mathrm{av}}(\Omega_\varepsilon)\big)^*, \\
\hat{\mathcal{E}}_\varepsilon^{(1)}(\mathbf{u}) &= \begin{cases} \frac{1}{2|\Omega_\varepsilon|} \displaystyle\int_{\Omega_\varepsilon} \mathbf{u}^2 \,\mathrm{d}\hat{x} & \text{if } \mathbf{u} \in \mathrm{L}^2(\Omega_\varepsilon), \\ \infty & \text{if } \mathbf{u} \in \big(\mathrm{H}^1_{\mathrm{av}}(\Omega_\varepsilon)\big)^* \setminus \mathrm{L}^2(\Omega_\varepsilon), \end{cases} \\
\hat{\mathcal{R}}_\varepsilon^{(1)*}(\boldsymbol{\xi}) &= \frac{|\Omega_\varepsilon|}{2} \int_{\Omega_\varepsilon} \nabla\boldsymbol{\xi} \cdot \mathrm{A}_\varepsilon(\hat{x}) \nabla\boldsymbol{\xi} \,\mathrm{d}\hat{x}
\end{aligned} \tag{4.2a}$$

with

$$\mathrm{H}^1_{\mathrm{av}}(\Omega_\varepsilon) = \Big\{ \xi \in \mathrm{H}^1(\Omega_\varepsilon) : \int_{\Omega_\varepsilon} \xi \,\mathrm{d}\hat{x} = 0 \Big\},$$

and the Boltzmann-Wasserstein-gradient system

$$\begin{aligned}
\mathbf{X}_\varepsilon^{(2)} &= \mathcal{P}(\overline{\Omega}_\varepsilon), \\
\hat{\mathcal{E}}_\varepsilon^{(2)}(\boldsymbol{\mu}) &= \begin{cases} \displaystyle\int_{\Omega_\varepsilon} \mathrm{E}_1\left(\frac{\mathrm{d}\boldsymbol{\mu}}{\mathrm{d}\boldsymbol{\pi}}\right) \mathrm{d}\boldsymbol{\pi} & \text{if } \boldsymbol{\mu} \ll \boldsymbol{\pi}, \\ \infty & \text{else}, \end{cases} \\
\hat{\mathcal{R}}_\varepsilon^{(2)*}(\boldsymbol{\mu}, \boldsymbol{\xi}) &= \frac{1}{2} \int_{\Omega_\varepsilon} \nabla\boldsymbol{\xi} \cdot \mathrm{A}_\varepsilon(\hat{x}) \nabla\boldsymbol{\xi} \,\mathrm{d}\boldsymbol{\mu}(\hat{x})
\end{aligned} \tag{4.2b}$$

where $\mathrm{E}_1(z) = z \log z - z + 1$ is the Boltzmann entropy. Note that the gradient flow equation induced by (4.2a) is (4.1) and for $\mathrm{d}\boldsymbol{\pi} = \mathrm{d}\hat{x}$ we have that (4.2b) induces (4.1) as well. We emphasize that the introduction of the linear tilt $\mathrm{D}\mathcal{E}_\varepsilon^{(j)} - \zeta$ corresponds to a change in the energies, i.e.,

$$\bar{\mathcal{E}}_\varepsilon^{(1)}(\mathbf{u}) = \frac{1}{2|\Omega_\varepsilon|} \int_{\Omega_\varepsilon} (\mathbf{u} - |\Omega_\varepsilon|\zeta)^2 \,\mathrm{d}\hat{x} \quad \text{satisfies } \mathrm{D}\bar{\mathcal{E}}_\varepsilon^{(1)}(\mathbf{u}) = \mathrm{D}\hat{\mathcal{E}}_\varepsilon^{(1)}(\mathbf{u}) - \zeta$$

and

$$\bar{\mathcal{E}}_\varepsilon^{(2)}(\boldsymbol{\mu}) = \int_{\Omega_\varepsilon} \mathrm{E}_1\left(\frac{\mathrm{d}\boldsymbol{\mu}}{\mathrm{d}\tilde{\boldsymbol{\pi}}}\right)\mathrm{d}\tilde{\boldsymbol{\pi}} \quad \text{with } \mathrm{d}\tilde{\boldsymbol{\pi}} = \mathrm{e}^{\zeta}\,\mathrm{d}\boldsymbol{\pi} \text{ satisfies } \mathrm{D}\bar{\mathcal{E}}_\varepsilon^{(1)}(\boldsymbol{\mu}) = \mathrm{D}\hat{\mathcal{E}}_\varepsilon^{(2)}(\boldsymbol{\mu}) - \zeta.$$

We observe that the tilts contribute differently to the equation depending on the choice of the gradient system, i.e.,

$$\dot{\mathbf{u}} = \partial\hat{\mathcal{R}}_\varepsilon^{(1)*}\big(-\mathrm{D}\bar{\mathcal{E}}_\varepsilon^{(1)}(\mathbf{u})\big) \Leftrightarrow \dot{\mathbf{u}} = \mathrm{div}\big(\mathrm{A}_\varepsilon(\nabla\mathbf{u} - \nabla\zeta)\big)$$

and

$$\dot{\boldsymbol{\mu}} = \partial\hat{\mathcal{R}}_\varepsilon^{(2)*}\big(-\mathrm{D}\bar{\mathcal{E}}_\varepsilon^{(2)}(\boldsymbol{\mu})\big) \Leftrightarrow \dot{\boldsymbol{\mu}} = \mathrm{div}\big(\mathrm{A}_\varepsilon(\nabla\boldsymbol{\mu} - \boldsymbol{\mu}\nabla\zeta)\big).$$

As shown below, the effective (dual) dissipation potential $\mathcal{R}_{\text{eff}}^{(1)*}$ is quadratic, whereas $\mathcal{R}_{\text{eff}}^{(2)*}$ is not. More precisely, $\mathcal{R}_{\text{eff}}^{(2)*}$ involves exponential terms. In other words, classical gradient systems may converge to generalized gradient systems. However the limit equation in both cases coincides and is linear.

We emphasize that our focus is on passing to the limit in the gradient system. The limiting equation is a consequence of the limiting gradient system. The analysis is done rigorously for the rescaled gradient systems where we have a fixed domain Ω_1. The ε-dependent domain Ω_ε is cylindric and given by $\Omega_\varepsilon = \Sigma \times (I_\varepsilon^- \cup \overline{I}_\varepsilon^0 \cup I_\varepsilon^+)$ with $\Sigma \subset \mathbb{R}^{d-1}$ and $I_\varepsilon^0 = (-\frac{\varepsilon^{1+\delta}}{2}, \frac{\varepsilon^{1+\delta}}{2}), I_\varepsilon^- = -\frac{\varepsilon^{1+\delta}}{2} + (-\varepsilon, 0)$ and $I_\varepsilon^+ = \frac{\varepsilon^{1+\delta}}{2} + (0,\varepsilon)$ where $\delta > 0$.

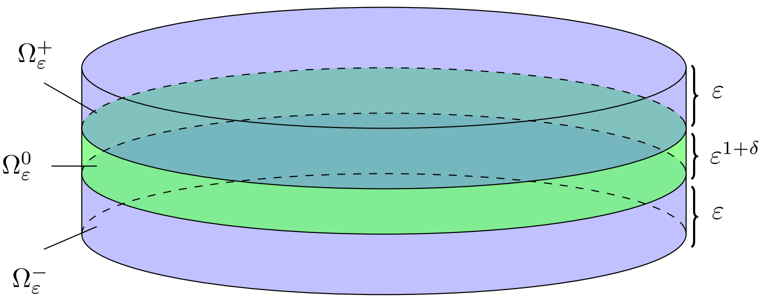

Figure 4.1: Domain Ω_ε

The coefficient matrix A_ε has a special block structure

$$\mathrm{A}_\varepsilon(\hat{x}) = \begin{pmatrix} \mathbf{B}_\varepsilon(\hat{x}) & 0 \\ 0 & \mathbf{a}_\varepsilon(\hat{x}) \end{pmatrix},$$

where $\mathbf{B}_\varepsilon \in \mathrm{L}^\infty(\Sigma;\mathbb{R}^{2\times2}_{\text{spd}})$ and $\mathbf{a}_\varepsilon \in \mathrm{L}^\infty(\Sigma;\mathbb{R}_{>0})$. Moreover, $\mathbf{B}_\varepsilon$ and $\mathbf{a}_\varepsilon$ have the following structure. Let $B \in \mathrm{L}^\infty(\Sigma;\mathbb{R}^{2\times2}_{\text{spd}})$ and $a \in \mathrm{L}^\infty(\Sigma;\mathbb{R}_{>0})$ with $\lambda, \Lambda \in \mathbb{R}$ such that

$$0 < \lambda \leq a, B \leq \Lambda < \infty. \tag{4.3}$$

Then with $\gamma \geq 0$ we have

$$\mathbf{B}_\varepsilon = \begin{cases} B \circ \Phi_\varepsilon^{-1} & \text{on } \Omega_1 \setminus \Omega_1^0, \\ \varepsilon^\gamma B \circ \Phi_\varepsilon^{-1} & \text{on } \Omega_1^0, \end{cases} \quad \text{and} \quad \mathbf{a}_\varepsilon(z) = \begin{cases} \varepsilon^{(2+\delta)} a \circ \Phi_\varepsilon^{-1} & \text{if } |z| < \varepsilon^\delta/2, \\ a \circ \Phi_\varepsilon^{-1} & \text{else.} \end{cases}$$

where $\Phi_\varepsilon : \Omega_\varepsilon \to \Omega_1$ is the piecewise affine homeomorphism depicted in Figure 4.2 with $\Phi(\Omega_\varepsilon^\iota) = \Omega_1^\iota$ and $\Phi_{|\Omega_\varepsilon^\iota}$ is affine for $\iota \in \{-, 0, +\}$. In particular, the special case from Chapter 2 $\mathrm{A}_\varepsilon = \mathbf{a}_\varepsilon I_d$ with $a \equiv 1$ is included, i.e., $\gamma = 2 + \delta$.

In the effective gradient structure it enters the harmonic mean on the middle layer which is denoted by

$$\mathrm{harm}_{I_1^0}(a) = \left(\int_{I_1^0} \frac{1}{a} \,\mathrm{d}z \right)^{-1}. \tag{4.4}$$

We denote $\big(\Phi_\varepsilon^{(1)}(\hat{x}), \dots, \Phi_\varepsilon^{(d-1)}(\hat{x})\big) = (\hat{x}_1, \dots, \hat{x}_{d-1}) = y \in \Sigma$ and $\Phi_\varepsilon^{(d)}(\hat{x}) = z \in I_1 := (I_1^- \cup \overline{I}_1^0 \cup I_1^+)$. More precisely, for $\hat{x} \in \Omega_\varepsilon^\pm$ we have

$$\Phi_\varepsilon^{(d)}(\hat{x}) = \begin{cases} \varepsilon^{-1}(\hat{x}_d - \varepsilon^{1+\delta} z^\pm) + z^\pm & \text{if } \hat{x} \in \Omega_\varepsilon^\pm \\ \varepsilon^{-(1+\delta)} \hat{x}_d & \text{if } \hat{x} \in \Omega_\varepsilon^0 \end{cases} \tag{4.5}$$

with $z^\pm = \pm\frac{1}{2}$.

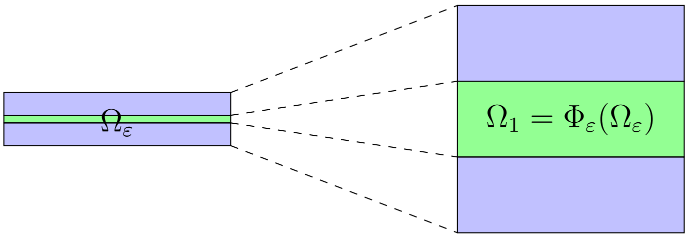

Figure 4.2: Rescaling Φ_ε

First, we perform the limit passage for (4.2a) in a standard Hilbert space setting where the Sandier-Serfaty approach [SS04] is applicable. In particular, we even have separate convergence of $\mathfrak{D}_\varepsilon^{\mathrm{dual}} \to \mathfrak{D}_{\mathrm{eff}}^{\mathrm{dual}}$ and $\mathfrak{D}_\varepsilon^{\mathrm{prim}} \to \mathfrak{D}_{\mathrm{eff}}^{\mathrm{prim}}$ with

$$\mathfrak{D}_\varepsilon^{\mathrm{dual}}(u) = \int_0^T \mathcal{R}_\varepsilon^*\big(-\mathrm{D}\mathcal{E}_\varepsilon(u)\big) \,\mathrm{d}t \tag{4.6a}$$

and

$$\mathfrak{D}_\varepsilon^{\mathrm{prim}}(\dot{u}) = \int_0^T \mathcal{R}_\varepsilon(\dot{u}) \,\mathrm{d}t. \tag{4.6b}$$

In constrast, the limit passage for (4.2b) is done via the methods introduced in Section 2.2 and features genuine EDP-convergence, i.e., we do not have the separate convergence, i.e., $\mathfrak{D}_\varepsilon^{\mathrm{dual}} \not\to \mathfrak{D}_{\mathrm{eff}}^{\mathrm{dual}}$ and $\mathfrak{D}_\varepsilon^{\mathrm{prim}} \not\to \mathfrak{D}_{\mathrm{eff}}^{\mathrm{prim}}$ but have to consider the convergence of the total dissipation $\mathfrak{D}_\varepsilon$.

4.1 The quadratic Hilbert-space setting

We transform the gradient system (4.2a) to $(\mathrm{X}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ by (formally) setting $u = \mathbf{m}_\varepsilon \mathbf{u} \circ \Phi_\varepsilon^{-1}$ and $\xi = \varepsilon \boldsymbol{\xi} \circ \Phi_\varepsilon^{-1}$ with

$$\mathbf{m}_\varepsilon(z) = \begin{cases} 1 & \text{if } z \in I^\pm \\ \varepsilon^\delta & \text{if } z \in I^0 \end{cases}.$$

Hence, the dual pairing reads for regular ξ, v

$$\langle \xi, v\rangle = \int_{\Omega_1} \xi v \,\mathrm{d}x = \int_{\Omega_\varepsilon} \boldsymbol{\xi}\ \mathbf{v} \,\mathrm{d}\hat{x}.$$

The transformation is rigorously defined by the map

$$\mathbf{E}^* : \mathbf{X}_\varepsilon^{(1)*} \ni \boldsymbol{\xi} \mapsto \varepsilon\boldsymbol{\xi} \circ \Phi_\varepsilon^{-1} \in \mathrm{X}^*.$$

We set $u = \mathbf{E}^{-1}\mathbf{u}$ where $\mathbf{E} : \mathrm{X} \ni u \mapsto \langle u, \mathbf{E}^*\cdot\rangle \in \mathbf{X}_\varepsilon^{(1)}$. For $\boldsymbol{\xi} \in \mathbf{X}^*$ we obtain that

$$\xi \in \mathrm{H}^1_{\mathrm{av}}(\Omega_1) = \Big\{\xi \in \mathrm{H}^1(\Omega_1) : \int_{\Omega_1} \xi \mathbf{m}_\varepsilon \,\mathrm{d}x = 0\Big\}.$$

The gradient system reads

$$\begin{aligned}
X &= \big(\mathrm{H}^1_{\mathrm{av}}(\Omega_1)\big)^*,\\
\mathcal{E}_\varepsilon(u) &:= \hat{\mathcal{E}}_\varepsilon^{(1)}(\mathbf{E}u) = \begin{cases} \frac{1}{2c_\varepsilon}\int_{\Omega_1} u^2 \frac{1}{\mathbf{m}_\varepsilon}\,\mathrm{d}x & \textbf{if } u \in \mathrm{L}^2(\Omega_1),\\ \infty & \textbf{if } u \in \big(\mathrm{H}^1_{\mathrm{av}}(\Omega_1)\big)^* \setminus \mathrm{L}^2(\Omega_1),\end{cases}\\
\mathcal{R}_\varepsilon(\xi) &:= \hat{\mathcal{R}}_\varepsilon^{(1)*}(\mathbf{E}^{-*}\xi) = \frac{c_\varepsilon}{2}\Big(\int_{\Omega_1} \mathbf{m}_\varepsilon \nabla_y \xi \cdot B_\varepsilon(x)\nabla_y \xi \,\mathrm{d}x\\
&\qquad + \int_{\Omega_1\setminus\Omega_1^0} \varepsilon^{-2} a(x)|\partial_z \xi|^2 \,\mathrm{d}x + \int_{\Omega_1^0} a(x)|\partial_z\xi|^2\,\mathrm{d}x\Big)
\end{aligned} \tag{4.7}$$

with $c_\varepsilon = (2+\varepsilon^\delta)|\Sigma|$. We recall that $x = (y, z)$. Hence, $\nabla_y = (\partial_1, \dots, \partial_{d-1})^T$. Moreover, the driving force reads

$$\mathrm{D}\mathcal{E}_\varepsilon(u) = \frac{1}{c_\varepsilon \mathbf{m}_\varepsilon} u.$$

The class of admissable tilts is given by $\{\zeta \in \mathrm{H}^1(\Omega_1) :\ (\partial_{\hat{z}}\zeta)_{|_{\Omega_1^\pm}} = 0\}$. The kinetic relation between rates $\dot{u}$ and forces $\xi_{\dot{u}}$ is given by

$$\forall \varphi \in X^* : \langle \varphi, \dot{u}\rangle = c_\varepsilon \int_{\Omega_1} \mathbf{m}_\varepsilon \nabla_y \xi_{\dot{u}} B \nabla_y \varphi + \varepsilon^{-2}\mathbf{m}_\varepsilon^{-1} a_\varepsilon \partial_z \xi_{\dot{u}} \partial_z \varphi \,\mathrm{d}x. \tag{4.8}$$

Note that $\mathbf{m}_{\varepsilon|_{\Omega_1^0}} \equiv \varepsilon^\delta$ and $(\varepsilon^{-(2+\delta}a_\varepsilon)_{|_{\Omega_1^0}} = a_{|_{\Omega_1^0}}$. Moreover, $\mathcal{R}_\varepsilon$ is given in terms of an Onsager operator $\mathbb{K}_\varepsilon$ defined for $\eta, \varphi \in X^*$ via

$$\langle \xi, \mathbb{K}_\varepsilon \eta\rangle = c_\varepsilon\Big(\int_{\Omega_1} \mathbf{m}_\varepsilon \nabla_y \xi \cdot B_\varepsilon(x)\nabla_y \eta \,\mathrm{d}x + \int_{\Omega_1\setminus\Omega_1^0} \varepsilon^{-2} a(x)\partial_z\xi\partial_z\eta\,\mathrm{d}x + \int_{\Omega_1^0} \mathrm{a}_0(x)\partial_z\xi\partial_z\eta\,\mathrm{d}x\Big).$$

Lemma 4.1. *We have indeed*

$$\hat{\mathcal{E}}_\varepsilon^{(1)}(\mathbf{E}u) = \begin{cases} \dfrac{1}{2c_\varepsilon}\displaystyle\int_{\Omega_1} u^2 \frac{1}{\mathbf{m}_\varepsilon}\,\mathrm{d}x & \textit{if } u \in \mathrm{L}^2(\Omega_1),\\ \infty & \textit{if } u \in \big(\mathrm{H}^1(\Omega_1)\big)^* \setminus \mathrm{L}^2(\Omega_1),\end{cases}$$

and

$$\hat{\mathcal{R}}_\varepsilon^{(1)*}(\mathbf{E}^{-*}\xi) = \frac{c_\varepsilon}{2}\Big(\int_{\Omega_1} \mathbf{m}_\varepsilon \nabla_y \xi \cdot B_\varepsilon(x)\nabla_y\xi \,\mathrm{d}x + \int_{\Omega_1\setminus\Omega_1^0} \varepsilon^{-2} a|\partial_z\xi|^2 \,\mathrm{d}x + \int_{\Omega_1^0} \mathrm{a}_0|\partial_z\xi|^2 \,\mathrm{d}x\Big).$$

Proof. Note that $u \in \mathrm{L}^2(\Omega_1) \Leftrightarrow \mathbf{E}u \in \mathrm{L}^2(\Omega_\varepsilon)$ and for $u \in \mathrm{L}^2(\Omega_1)$ we compute $\frac{u}{\mathbf{m}_\varepsilon}\circ\Phi_\varepsilon = \mathbf{E}u$ and

$$\int_{\Omega_\varepsilon} (\mathbf{E}u)^2 \,\mathrm{d}\hat{x} = \int_{\Omega_1} \varepsilon \mathbf{m}_\varepsilon \left(\frac{u}{\mathbf{m}_\varepsilon}\right)^2 \mathrm{d}x.$$

Moreover, the observation $\varepsilon c_\varepsilon = |\Omega_\varepsilon|$ concludes the formula for $\mathcal{E}_\varepsilon(u)$. Regarding the dissipation potential, we have $\boldsymbol{\xi} := \mathbf{E}^{-*}\xi = \frac{\xi}{\varepsilon}\circ\Phi_\varepsilon$. Hence, we compute

$$\begin{aligned}
|\Omega_\varepsilon| \int_{\Omega_\varepsilon} \nabla_{\hat{y}}\boldsymbol{\xi} B_\varepsilon \nabla_{\hat{y}}\boldsymbol{\xi} \,\mathrm{d}\hat{x} &\overset{\text{(i)}}{=} |\Omega_\varepsilon| \int_{\Omega_\varepsilon} \varepsilon \mathbf{m}_\varepsilon \nabla_y \frac{\xi}{\varepsilon} B_\varepsilon \nabla_y \frac{\xi}{\varepsilon} \,\mathrm{d}x = c_\varepsilon \int_{\Omega_\varepsilon} \mathbf{m}_\varepsilon \nabla_y \xi \mathbf{M}_\varepsilon \nabla_y \xi \,\mathrm{d}x,\\
|\Omega_\varepsilon| \int_{\Omega_\varepsilon\setminus\Omega_\varepsilon^0} a_\varepsilon(\hat{x})|\partial_{\hat{z}}\boldsymbol{\xi}|^2 \,\mathrm{d}\hat{x} &\overset{\text{(ii)}}{=} |\Omega_\varepsilon| \int_{\Omega_1\setminus\Omega_1^0} \varepsilon a_\varepsilon(x) \left|\partial_z \frac{\xi}{\varepsilon}\right|^2 \varepsilon^{-2} \,\mathrm{d}x = c_\varepsilon \varepsilon^{-2} \int_{\Omega_1\setminus\Omega_1^0} a_\varepsilon(x)|\partial_z\xi|^2 \,\mathrm{d}x,\\
|\Omega_\varepsilon| \int_{\Omega_\varepsilon^0} \mathbf{a}_\varepsilon(\hat{x})|\partial_{\hat{z}}\boldsymbol{\xi}|^2 \,\mathrm{d}\hat{x} &\overset{\text{(iii)}}{=} |\Omega_\varepsilon| \int_{\Omega_1^0} \varepsilon^{-(3+\delta)} a_\varepsilon(x) \,|\partial_z\xi|^2 \,\mathrm{d}x \overset{\text{(iv)}}{=} c_\varepsilon \int_{\Omega_1^0} a(x)|\partial_z\xi|^2 \,\mathrm{d}x,
\end{aligned}$$

where we used (i) $\partial_{\hat{y}}\Phi_\varepsilon(\hat{x}) \equiv I_{d-1}$, (ii) $(\partial_{\hat{z}}\Phi_\varepsilon)_{|\Omega_\varepsilon\setminus\Omega_\varepsilon^0} \equiv \varepsilon^{-1}$, (iii) $(\partial_{\hat{z}}\Phi_\varepsilon)_{|\Omega_\varepsilon^0} \equiv \varepsilon^{-(1+\delta)}$ and (iv) $a_{\varepsilon|\Omega_1^0} = \varepsilon^{2+\delta} a_{|\Omega_1^0}$. □

4.1.1 Compactness

In the following we derive for general curves $u_\varepsilon : [0,T] \to \mathrm{X}$ with

$$\sup_{t,\varepsilon} \mathcal{E}_\varepsilon\big(u_\varepsilon(t)\big) < \infty \quad \text{and} \quad \sup_\varepsilon \mathfrak{D}_\varepsilon(u_\varepsilon) < \infty \tag{4.9}$$

compactness results with respect to certain topologies. Note that solutions to the gradient flow equation satisfy (4.9). The topologies for the Γ-limits of $\mathcal{E}_\varepsilon$ and $\mathfrak{D}_\varepsilon$ are chosen such that $u_\varepsilon(t)$ is relatively compact for all $t \in [0,T]$ and that u_ε is relatively compact respectively.

Lemma 4.2. *Let $u_\varepsilon \in \mathrm{X}$ be such that $\sup_\varepsilon \mathcal{E}_\varepsilon(u_\varepsilon) < \infty$ then*

$$\sup_\varepsilon \|u_\varepsilon\|_{\mathrm{L}^2(\Omega_1^\pm)} < \infty \quad \textit{and} \quad u_{\varepsilon|\Omega_1^0} \to 0 \quad \textit{in}\ \mathrm{L}^2(\Omega_1^0).$$

Proof. From the definition of $\mathcal{E}_\varepsilon$ follows that

$$\mathcal{E}_\varepsilon(u) = 1/(2c_\varepsilon)(\|u_\varepsilon\|^2_{\mathrm{L}^2(\Omega_1\setminus\Omega_1^0)} + \varepsilon^{-\delta}\|u_\varepsilon\|^2_{\mathrm{L}^2(\Omega_1\setminus\Omega_1^0)})$$

and the claim follows. □

In particular, the conditions of Lemma 4.2 give weak compactness of u_ε in $\mathrm{L}^2(\Omega_1)$. To derive compactness of curves $u_\varepsilon : [0,T] \to \mathrm{X}$, we introduce the space $\mathbb{X}^* := \mathrm{H}^1(\Sigma) \times \mathrm{H}^1(\Sigma)$ and the operator $\mathbb{L} : \mathbb{X}^* \to \mathrm{H}^1(\Omega_1)$ with

$$\mathbb{L}[\eta](y,z) = \eta^+(y)p_+(z) + \eta^-(y)p_-(z),$$

where for $z \in [z_-, z_+]$

$$p_+(z) = \frac{z - z_-}{z_+ - z_-} \text{ and } p_+(z) = \frac{z_+ - z}{z_+ - z_-} \tag{4.10}$$

and $p_\pm(z) = p_\pm(z^\pm) \in \{0,1\}$ for $\pm z > \pm z_\pm$. The operator $\mathbb{L}$ allows us to embed $\mathrm{L}^2(\Sigma) \times \mathrm{L}^2(\Sigma)$ into $\mathrm{L}^2(\Omega_1)$. It is obvious that there exists $0 < c < C < \infty$ such that

$$c\|\eta\|^2_{\mathbb{X}^*} \leq \langle \mathbb{L}\eta, \mathbb{K}_\varepsilon \mathbb{L}\eta \rangle \leq C\|\eta\|^2_{\mathbb{X}^*} \tag{4.11}$$

where

$$\|\eta\|^2_{\mathbb{X}^*} = \|\nabla\eta^-\|^2_{\mathrm{L}^2(\Sigma)} + \|\nabla\eta^+\|^2_{\mathrm{L}^2(\Sigma)} + \|[\![\eta]\!]\|^2_{\mathrm{L}^2(\Sigma)} \text{ and } [\![\eta]\!] = \eta^+ - \eta^-.$$

The semi norm $\|\cdot\|_{\mathbb{X}^*}$ defines a norm on

$$\mathbb{X}^*_{\mathrm{av}} = \mathbb{X}^* \cap \left\{ (\eta^+, \eta^-) \in \mathrm{L}^2(\Sigma) \times \mathrm{L}^2(\Sigma) \,\middle|\, \int_\Sigma \eta^+ + \eta^- \,\mathrm{d}y = 0 \right\}.$$

The next lemma states the compactness result for curves $\mathbb{L}^* u_\varepsilon : [0,T] \to \mathbb{X}$ since we can only use test-functions satisfying $(\partial_z \xi)_{|_{\Omega_1 \setminus \Omega_1^0}} \equiv 0$. Here it is sufficient to consider test functions ξ of the form $\xi = \mathbb{L}\eta$. For $v \in \mathrm{L}^2(\Omega_1)$ we easily compute

$$\mathbb{L}^* v = \left(\int_I v p_-(z) \,\mathrm{d}z, \int_I v p_+(z) \,\mathrm{d}z \right) \quad \text{and} \quad \int_\Sigma (\mathbb{L}^* v)^- + (\mathbb{L}^* v)^+ \,\mathrm{d}y = \int_{\Omega_1} v \,\mathrm{d}x.$$

In particular, $\mathbb{L}^* u_\varepsilon$ is a reduced object acting only twice in the cross-section Σ which corresponds to the upper and lower layer. Moreover, we need convergence pointwise in time, to pass to the limit in $\mathcal{E}_\varepsilon\big(u_\varepsilon(t)\big)$ which is justified by an Aubin-Lion lemma.

Lemma 4.3. *Let u_ε such that*

$$\sup_\varepsilon \left\{ \mathfrak{D}_\varepsilon(u_\varepsilon) + \sup_t \mathcal{E}_\varepsilon\big(u_\varepsilon(t)\big) \right\} < \infty.$$

Then there exists a limit function $u \in \mathrm{C}^0(0,T;\mathbb{X}) \cap \mathrm{L}^\infty\big(0,T;\mathrm{L}^2(\Sigma)^2\big)$ and a subsequence such that

$$\mathbb{L}^* u_\varepsilon \to u \text{ in } \mathrm{C}^0(0,T;\mathbb{X}) \text{ and } \forall\, t \in [0,T] : \ \mathbb{L}^* u_\varepsilon(t) \rightharpoonup u(t) \in \mathrm{L}^2(\Sigma) \times \mathrm{L}^2(\Sigma).$$

Proof. With $\mathbb{G}_\varepsilon = \mathbb{K}_\varepsilon^{-1}$ we use the Cauchy-Schwarz estimate

$$\int_{t_1}^{t_2} \langle \xi, \dot u\rangle \,\mathrm{d}t \le \left(\int_{t_1}^{t_2} \langle \xi, \mathbb{K}_\varepsilon \xi\rangle \,\mathrm{d}t\right)^{1/2} \left(\int_{t_1}^{t_2} \langle \mathbb{G}_\varepsilon \dot u, \dot u\rangle \,\mathrm{d}t\right)^{1/2},$$

the norm equivalence (4.11) and the bound on $\int_0^T \mathcal{R}_\varepsilon(\dot u_\varepsilon)\,\mathrm{d}t$ to obtain the equicontinuity

$$\sup_{\substack{\eta\in\mathbb{X}_{\mathrm{av}}^* \\ \|\eta\|_{\mathbb{X}^*}\le 1}} \langle \mathbb{L}\eta, u_\varepsilon(t_1)-u_\varepsilon(t_2)\rangle =: \|\mathbb{L}^*\big(u_\varepsilon(t_1)-u_\varepsilon(t_2)\big)\|_{\mathbb{X}_{\mathrm{av}}} \le C\sqrt{|t_1-t_2|}.$$

The map $\mathbb{X}_{\mathrm{av}} \ni v \mapsto \{\eta \mapsto \langle v, \eta - \mathcal{A}\eta\rangle\} \in \mathbb{X}$ defines a continuous embedding $\mathbb{X}_{\mathrm{av}} \hookrightarrow \mathbb{X}$, where $\mathcal{A}\eta = \int_\Sigma \eta^- + \eta^+ \,\mathrm{d}y$. Moreover, by Lemma 4.2 and the compact embedding $\mathrm{L}^2(\Sigma)\times\mathrm{L}^2(\Sigma) \hookrightarrow \mathbb{X}_{\mathrm{av}}$ we have strong compactness of $\mathbb{L}^* u_\varepsilon(t)$ in $\mathbb{X}_{\mathrm{av}}$ and hence, in $\mathbb{X}$. Applying [Sim87, Cor. 4] finishes the proof. $\square$

The following lemma serves in the sequel as a compactness result for the driving force $\mathrm{D}\mathcal{E}_\varepsilon(u_\varepsilon)$ as well as for the solution ξ_ε to the kinetic relation (4.8).

Lemma 4.4. *Let* $(\xi_\varepsilon)_{\varepsilon>0} \subset \mathrm{L}^2(0,T;\mathrm{X}^*)$ *satisfy*

$$\sup_{\varepsilon>0}\left\{\int_0^T \|\nabla_y \xi_\varepsilon(t)\|^2_{\mathrm{L}^2(\Omega_1\setminus\Omega_1^0)} + \|\partial_z \xi_\varepsilon(t)\|^2_{\mathrm{L}^2(\Omega_1^0)} + \|\varepsilon^{-1}\partial_z \xi_\varepsilon(t)\|^2_{\mathrm{L}^2(\Omega_1\setminus\Omega_1^0)}\right\}\mathrm{d}t < \infty.$$

Then there exists $\eta \in \mathrm{L}^2(0,T;\mathbb{X}_{\mathrm{av}}^*)$ *and a subsequence such that*

$$\xi_\varepsilon|_{\Omega_1^\pm} \rightharpoonup \eta^\pm_{\mathrm{ext}} \text{ in } \mathrm{L}^2\big(0,T;\mathrm{H}^1(\Omega_1^\pm)\big),$$

with $\eta^\pm_{\mathrm{ext}}(y,z) = \eta^\pm(y)$. *Moreover, we also have relatively weak compactness of* $\xi_\varepsilon|_{\Omega_1^0}$ *in* $\mathrm{L}^2\big(0,T;\mathrm{L}^2(\Omega_1^0)\big)$.

Proof. Since the gradient $(\nabla\xi_\varepsilon)|_{\Omega_1\setminus\Omega_1^0}$ is bounded by assumptions it remains to show an L^2 bound for ξ_ε. For this, we use the Poincaré-Wirtinger estimate for a (regular) domain $\mathcal{D}$ and $\xi \in \mathrm{H}^1(\mathcal{D})$ with its integral mean $\xi_{\mathcal{D}} := \fint_{\mathcal{D}} \xi \,\mathrm{d}x$

$$\|\xi - \xi_{\mathcal{D}}\|_{\mathrm{L}^2(\mathcal{D})} \le C(\mathcal{D})\|\nabla\xi\|_{\mathrm{L}^2(\mathcal{D})}$$

to deduce the fact that ξ_ε is bounded in $\mathrm{L}^2\big(0,T;\mathrm{L}^2(\Omega_1\setminus\Omega_1^0)\big)$ if and only if we have $\sup_\varepsilon \int_0^T |\fint_{\Omega_1^+}\xi_\varepsilon\,\mathrm{d}x|^2 + |\fint_{\Omega_1^-}\xi_\varepsilon\,\mathrm{d}x|^2\,\mathrm{d}t < \infty$.

We consider the averaged quantities

$$\sigma_\varepsilon(x_3) = \fint_\Sigma \xi_\varepsilon(x)\,\mathrm{d}y \text{ and their means } \sigma_\varepsilon^\nu = \fint_{I^\nu} \sigma_\varepsilon\,\mathrm{d}x_3 \text{ with } \nu\in\{+,-,0\}.$$

hence, we have to show that σ_ε^+ and σ_ε^- are bounded in $\mathrm{L}^2(0,T)$. By assumption we have $\int_{\Omega_1} \mathbf{m}_\varepsilon \xi_\varepsilon\,\mathrm{d}x = 0$ and thus we have $\sigma_\varepsilon^+ + \varepsilon^\delta\sigma_\varepsilon^0 + \sigma_\varepsilon^- = 0$ a.e. We show both

$$\text{(i)} \sup_\varepsilon \|\sigma_\varepsilon^+ - \sigma_\varepsilon^-\|_{\mathrm{L}^2(0,T)} < \infty \quad\text{and}\quad \text{(ii)} \sup_\varepsilon \|\sigma_\varepsilon^+ + \sigma_\varepsilon^-\|_{\mathrm{L}^2(0,T)} < \infty.$$

We easily see (i) since $\|\partial_z \sigma_\varepsilon\|_{L^2([0,T]\times I_1)}$ by the bound on $\|\partial_z \xi_\varepsilon\|_{L^2([0,T]\times\Omega_1)}$. For (ii) we use the identities $\varepsilon^\delta \sigma_\varepsilon^0 = -\sigma_\varepsilon^+ - \sigma_\varepsilon^-$,

$$\sigma_\varepsilon(z) = \sigma_\varepsilon(z_+) - \int_z^{z_+} \partial_z \sigma_\varepsilon \, \mathrm{d}x_3 = \sigma_\varepsilon(z_-) + \int_{z_-}^{z} \partial_z \sigma_\varepsilon \, \mathrm{d}x_3.$$

and

$$\begin{aligned}(2+\varepsilon^\delta)\sigma_\varepsilon^0 &= (\sigma_\varepsilon(z_+) - \sigma_\varepsilon^+) + (\sigma_\varepsilon(z_-) - \sigma_\varepsilon^-) \\ &\quad + \int_{z_-}^{z_+} \left(\int_{z_-}^{z} \partial_z \sigma_\varepsilon \, \mathrm{d}x_3 - \int_z^{z_+} \partial_z \sigma_\varepsilon \, \mathrm{d}x_3 \right) \mathrm{d}z \end{aligned} \tag{4.12}$$

to conclude with Jensen's estimate

$$\sqrt{|\Omega_1^\pm|}\, |\sigma_\varepsilon^\pm - \sigma_\varepsilon(z_\pm)| \le \|\xi_\varepsilon - \xi_\varepsilon(\cdot, z_\pm)\|_{L^2(\Omega_1^\pm)} \le \|\partial_z \xi_\varepsilon\|_{L^2(\Omega_1^\pm)} \longrightarrow 0 \text{ in } L^2(0,T).$$

hence, we get a weakly convergent subsequence $\xi_\varepsilon \rightharpoonup \eta_{\mathrm{ext}}^\pm$ in $L^2\big(0,T;H^1(\Omega_1^\pm)\big)$ with $\int_\Sigma \eta^+(t) + \eta^-(t)\,\mathrm{d}y = 0$.

Moreover, we find that $\xi_\varepsilon|_{\Omega_1^0}$ is bounded in $L^2\big(0,T;L^2(\Omega_1^0)\big)$. □

4.1.2 The Γ-liminf estimate of $\mathfrak{D}_\varepsilon$

In the sequel, we pass to the Γ-liminf seperately for $\mathfrak{D}_\varepsilon^{\mathrm{dual}}$ as well as $\mathfrak{D}_\varepsilon^{\mathrm{prim}}$ from (4.6). The following lemma is used for both since the primal dissipation potential can be expressed via the dual dissipation potential using the kinetic relation (4.8), i.e., $\mathcal{R}_\varepsilon(\dot u_\varepsilon) = \mathcal{R}_\varepsilon^*(\xi_\varepsilon)$.

Lemma 4.5. *Let* $\xi_\varepsilon|_{\Omega_1^\pm} \rightharpoonup \eta^\pm$ *in* $L^2\big(0,T;H^1(\Omega_1^\pm)\big)$. *Then we have*

$$\liminf_{\varepsilon\downarrow 0} \int_0^T \mathcal{R}_\varepsilon^*(\xi_\varepsilon)\,\mathrm{d}t \ge \int_0^T \mathcal{R}_{\mathrm{eff}}^*(\eta)\,\mathrm{d}t$$

with

$$\mathcal{R}_{\mathrm{eff}}^*(\eta) = \begin{cases} \dfrac{c_0}{2} \displaystyle\int_\Sigma \Big(\sum_{\iota\in\{-,+\}} \nabla\eta^\iota B^\iota \nabla\eta^\iota \Big) + \dfrac{c_0}{2}\, \mathrm{harm}_{I_1^0}(a)\, [\![\eta]\!]^2 \,\mathrm{d}y & \textit{if } \partial_z \eta^\pm \equiv 0 \\ \infty & \textit{else.} \end{cases}$$

with $B^\pm = \int_{I^\pm} B\,\mathrm{d}z$ *and* $\mathrm{harm}_{I^0}(a)$ *defined in* (4.4).

Proof. Without loss of generality we assume $\liminf_{\varepsilon\downarrow 0} \int_0^T \mathcal{R}_\varepsilon^*(\xi_\varepsilon)\,\mathrm{d}t < \infty$. Hence, Lemma 4.4 applies and we find $\partial_z \eta^\pm = 0$. By dropping positive terms we estimate

$$\mathcal{R}_\varepsilon^*(\xi_\varepsilon) \ge \sum_{\iota\in\{-,+\}} \frac{c_\varepsilon}{2} \int_{\Omega_1^\iota} \nabla_y \xi_\varepsilon B \nabla_y \xi_\varepsilon \,\mathrm{d}x + \frac{c_\varepsilon}{2} \int_{\Omega_1^0} a |\partial_z \xi_\varepsilon|^2 \,\mathrm{d}x.$$

Moreover, Jensen's estimate with respect to the probability measure $\mathbb{P}$ on I_1^0 given by $\mathrm{d}\mathbb{P} = \mathrm{harm}_{I_1^0}(a)\,\frac{1}{a}\,\mathrm{d}z$ yields

$$\int_{I_1^0} a|\partial_z\xi|^2\,\mathrm{d}z \geq \mathrm{harm}_{I_1^0}(a)\,[\![\xi]\!]^2.$$

Hence, by weak convergence and convexity the claim is proved. □

In order to exploit Lemma 4.5 for the limit passage of $\mathfrak{D}_\varepsilon^{\mathrm{prim}}$ we need to know the limiting kinetic relation between rates and forces. Since u_ε vanishes in the middle layer we obtain a contribution of the jump between the upper and lower layer.

Lemma 4.6. *Let* $\{u_\varepsilon\} \subset \mathrm{X}$ *be such that*

$$\mathbb{L}^* u_\varepsilon \to u \textit{ in } \mathrm{C}^0(0,T;\mathbb{X}) \textit{ and } \forall\, t \in [0,T]: \;\; \mathbb{L}^* u_\varepsilon(t) \rightharpoonup u(t) \in \mathrm{L}^2(\Sigma) \times \mathrm{L}^2(\Sigma)$$

and

$$\sup_\varepsilon \int_0^T \mathcal{R}_\varepsilon(\dot{u}_\varepsilon)\,\mathrm{d}t < \infty.$$

Let ξ_ε *be the solution to the kinetic relation* (4.8). *Then we have weak convergence* $\xi_\varepsilon|_{\Omega_1^\pm} \rightharpoonup \xi_{\mathrm{ext}}^\pm$ *in* $\mathrm{L}^2\big(0,T;\mathrm{H}^1(\Omega_1^\pm)\big)$ *with* $\xi \in \mathbb{X}^*$ *satisfying*

$$\forall\, \varphi \in \mathbb{X}_{\mathrm{av}}^* \quad \langle \varphi, \dot{u}\rangle = c_0 \int_\Sigma \big(\sum_{\iota\in\{-,+\}} \nabla\xi^\iota B^\iota \nabla\varphi^\iota\big) + c_0\,\mathrm{harm}\,(a_0)\,[\![\xi]\!][\![\varphi]\!]\,\mathrm{d}y, \tag{4.13}$$

i.e., the kinetic relation is given by an Onsager operator $\mathbb{K}_0 : \mathbb{X} \to \mathbb{X}^*$. *In particular, for all* $\varphi \in \mathrm{L}^2\big(0,T;\mathrm{H}^1(\Omega_1)\big)$ *such that* $(\partial_z\varphi)|_{\Omega_1\setminus\Omega_1^0} = 0$ *we have*

$$\int_0^T \langle \varphi, \mathbb{K}_\varepsilon \xi_\varepsilon\rangle\,\mathrm{d}t \to \int_0^T \langle \varphi, \mathbb{K}_0 \xi\rangle\,\mathrm{d}t.$$

Proof. Since ξ_ε satisfies the assumptions of Lemma 4.4 we deduce a limit $\xi \in \mathbb{X}_{\mathrm{av}}^*$ of $\xi_\varepsilon|_{\Omega_1\setminus\Omega_1^0}$ with

$$\langle \varphi, \mathbb{K}_\varepsilon\xi_\varepsilon\rangle \to c_0 \int_\Sigma \big(\sum_{\iota\in\{-,+\}} \nabla\varphi^\iota B^\iota \nabla\xi^\iota\big)\,\mathrm{d}y + c_0 \int_{\Omega_1^0} a\partial_z\tilde{\xi}\partial_z\varphi\,\mathrm{d}x$$

for every $\varphi \in \mathrm{L}^2\big(0,T;\mathrm{H}^1(\Omega_1)\big)$ such that $(\partial_z\varphi)_{\Omega_1^\pm} = 0$. Where $\tilde{\xi}$ is the weak limit of $\xi_\varepsilon|_{\Omega_1^0}$, which exists due to Lemma 4.4. On the other hand, testing with $\varphi \in \mathrm{C}^1([0,T]\times\Omega_1)$ we obtain

$$\int_0^T \langle \varphi, \dot{u}_\varepsilon\rangle\,\mathrm{d}t \to \int_0^T \langle \varphi, \dot{u}\rangle\,\mathrm{d}t$$

in the sense of distributions. In particular, for φ such that for all t we have $\operatorname{supp}\big(\varphi(t)\big) \subset \Omega_1^0$ we obtain $\int_0^T \langle \varphi, \dot u\rangle \,\mathrm{d}t = 0$ since $u_{\varepsilon}|_{\Omega_1^0} \to 0$. Hence

$$\forall\, \varphi \in \mathrm{C}_0^1([0,T]\times\Omega_1^0) \qquad \int_0^T \int_{\Omega_1^0} a_0 \partial_z \tilde\xi \partial_z \varphi \,\mathrm{d}x \,\mathrm{d}t = 0.$$

Thus we conclude $\partial_z(a\partial_z\tilde\xi) = 0$ which is explicitly solvable and we obtain $a\partial_z\tilde\xi = \operatorname{harm}_{I_1^0}(a)\,[\![\xi]\!]$ since $[\![\xi]\!] = [\![\tilde\xi]\!]$. By continuity we extend the equality

$$\int_0^T \langle \varphi, \dot u\rangle \,\mathrm{d}t = c_0 \int_\Sigma \Big(\sum_{\iota\in\{-,+\}} \nabla\xi^\iota B^\iota \nabla\varphi^\iota \Big) \,\mathrm{d}y + c_0 \operatorname{harm}(a_0)\,[\![\xi]\!][\![\varphi]\!] \,\mathrm{d}x$$

for all $\varphi \in \mathrm{L}^2(0,T;\mathbb{X}^*)$. ◻

Combining Lemma 4.5 and Lemma 4.6 we conclude the following Γ-liminf estimate for the tilted total dissipation functional

$$\mathfrak{D}_\varepsilon^\zeta(u) = \int_0^T \mathcal{R}_\varepsilon(\dot u) + \mathcal{R}_\varepsilon^*\big(-\mathrm{D}\mathcal{E}_\varepsilon(u) + \zeta\big) \,\mathrm{d}t.$$

Theorem 4.7. *Let $\{u_\varepsilon\} \subset \mathrm{X}$ be such that*

$$\mathbb{L}^* u_\varepsilon \to u \text{ in } \mathrm{C}^0(0,T;\mathbb{X}) \text{ and } \forall\, t\in[0,T]:\ \ \mathbb{L}^* u_\varepsilon(t) \rightharpoonup u(t) \in \mathrm{L}^2(\Sigma)^2.$$

Then

$$\liminf \mathfrak{D}_\varepsilon^\zeta(u_\varepsilon) \geq \mathfrak{D}_{\mathrm{eff}}^\zeta(u) = \int_0^T \mathcal{R}_{\mathrm{eff}}(\dot u) + \mathcal{R}_{\mathrm{eff}}^*\big(-\mathrm{D}\mathcal{E}_0(u) + \zeta\big) \,\mathrm{d}t.$$

Proof. Lemma 4.4 gives weak convergence of $(-\mathrm{D}\mathcal{E}_\varepsilon(u)+\zeta)|_{\Omega_1^\pm}$ and $\xi_\varepsilon|_{\Omega_1^\pm}$ in the Hilbert space $\mathrm{L}^2\big(0,T;\mathrm{H}^1(\Omega_1^\pm)\big)$ where ξ_ε is the solution to the kinetic relation (4.8). Hence, we apply Lemma 4.5 and Lemma 4.6 to conclude the proof. ◻

A straight forward computation gives that $\mathcal{E}_\varepsilon \xrightarrow{\mathrm{M}} \mathcal{E}_0$ in Mosco-sense with respect to $\mathrm{L}^2(\Omega_1)$ where

$$\mathcal{E}_0(u) = \frac{1}{2c_0}\int_{\Omega_1^-} u^2 \,\mathrm{d}x + \frac{1}{2c_0}\int_{\Omega_1^+} u^2 \,\mathrm{d}x$$

if $u|_{\Omega_1^0} = 0$, and $\mathcal{E}_0(u) = \infty$ else. In particular, for u such that $(\partial_z u)|_{\Omega_1^\pm} = 0$ we have with $u|_{\Omega_1^\pm} =: u^\pm$ the identity

$$\mathcal{E}_0(u) = \frac{1}{2c_0}\int_\Sigma |u^-|^2 + |u^+|^2 \,\mathrm{d}y.$$

4.1.3 The Γ-limsup estimate of $\mathfrak{D}_\varepsilon$

The recovery sequence for $\mathfrak{D}^{\zeta}_{\text{eff}}$ is constructed as follows. Let $u^\pm \in \mathrm{L}^2(\Sigma)$ be arbitrary. For $z \in [z^-, z^+]$ we define

$$p_a^+(z) := \operatorname{harm}_{I_1^0}(a) \int_{z^-}^{z} \frac{1}{a}\,\mathrm{d}z' \quad \text{and} \quad p_a^-(z) := \operatorname{harm}_{I_1^0}(a) \int_{z}^{z^+} \frac{1}{a}\,\mathrm{d}z' \tag{4.14}$$

and the lineare operator

$$\mathfrak{r}_\varepsilon u = \begin{cases} u^+ & \text{in } \Omega_1^+ \\ \varepsilon^\delta (p_a^+ u^+ + p_a^- u^-) & \text{in } \Omega_1^0 \\ u^- & \text{in } \Omega_1^-. \end{cases}$$

The tilt ζ introduces the shift

$$\mathfrak{s}_\varepsilon \zeta : [z^-, z^+] \ni z \mapsto \varepsilon^\delta(\zeta_\varepsilon(z) - \zeta_\varepsilon(z^+))p_a^+(z) + \varepsilon^\delta(\zeta_\varepsilon(z) - \zeta_\varepsilon(z^-))p_a^-(z).$$

Note that $\lim_{h\downarrow 0} \mathfrak{r}_\varepsilon u(y, z^\pm \mp h) = \varepsilon^\delta u^\pm(y)$ and $\mathfrak{s}_\varepsilon\zeta(z^\pm) = 0$. Hence, we extend $\mathfrak{s}_\varepsilon\zeta$ trivially to the whole interval I_1. Moreover, for $z \in I_1^0$ we have

$$a\partial_z(\mathfrak{r}_\varepsilon u + \mathfrak{s}_\varepsilon\zeta) \equiv \varepsilon^\delta \operatorname{harm}_{I_1^0}(a)\, [\![u - \zeta_\varepsilon]\!].$$

The recovery operator is then defined by

$$u_\varepsilon = \mathfrak{r}_\varepsilon u + \mathfrak{s}_\varepsilon \zeta.$$

We denote by $U = \varepsilon^{-\delta}(\mathfrak{r}_\varepsilon u + \mathfrak{s}_\varepsilon\zeta)$ the optimal profile on the middle layer. For simplicity we assume that $a_{|\Omega_1^0} \in \mathrm{W}^{1,\infty}(\Omega_1^0)$ which gives the regularity $U \in \mathrm{H}^1(\Omega_1^0)$ if $u^\pm \in \mathrm{H}^1(\Omega_1^\pm)$.

To prove the Γ-limsup we need to estimate $\mathbb{K}_0 \lesssim \mathbb{K}_\varepsilon$. Then we are able to conclude that $\mathfrak{r}_\varepsilon u$ is sufficiently regular in time, i.e., $\mathfrak{D}^{\text{prim}}_\varepsilon(\mathfrak{r}_\varepsilon u)$ is bounded.

Theorem 4.8. *Let* $u \in \mathrm{L}^\infty(0,T; \mathrm{L}^2(\Sigma) \times \mathrm{L}^2(\Sigma)) \cap \mathrm{C}^0([0,T]; \mathbb{X})$, $a_{|\Omega_1^0} \in \mathrm{W}^{1,\infty}(\Omega_1^0)$ *and* $\zeta \in \mathrm{H}^1(\Omega_1)$ *such that* $\partial_z\zeta \equiv 0$ *in* $\Omega_1 \setminus \Omega_1^0$. *Then the recovery operator satisfies*

$$\lim_{\varepsilon\to 0} \mathfrak{D}^\zeta_\varepsilon(\mathfrak{r}_\varepsilon u) = \mathfrak{D}^\zeta_{\text{eff}}(u).$$

Proof. Without loss of generality we assume $\mathfrak{D}_{\text{eff}}(u) < \infty$. We show convergence of the dual part $\mathfrak{D}^{\zeta,\text{dual}}_\varepsilon$ and the primal part $\mathfrak{D}^{\text{prim}}_\varepsilon$ separately. We abbreviate $q_B : \mathbb{R}^{d-1} \ni v \mapsto v \cdot Bv \in \mathbb{R}$ and compute

$$\begin{aligned} \mathcal{R}^*_\varepsilon\big(-\mathrm{D}\mathcal{E}_\varepsilon(\mathfrak{r}_\varepsilon u) + \zeta\big) = \frac{c_\varepsilon}{2} \sum_{\iota \in \{-,+\}} \int_\Sigma q_B\Big(\nabla_y\big(c_\varepsilon^{-1}u^\iota - \zeta^\iota\big)\Big)\,\mathrm{d}y \\ + \frac{\varepsilon^\delta c_\varepsilon}{2} \int_{\Omega_1^0} q_B\big(\nabla_y(c_\varepsilon^{-1}U - \zeta)\big)\,\mathrm{d}x \\ + \frac{c_\varepsilon}{2} \int_{\Omega_1^0} a^{-1}\big|\operatorname{harm}_{I_1^0}(a)\, [\![c_\varepsilon^{-1}u - \zeta]\!]\big|^2\,\mathrm{d}x \end{aligned}$$

By regularity of a and ζ we obtain that $\nabla_y U \in \mathrm{L}^2\big((0,T)\times\Omega_1^0\big)$ and

$$\lim \int_0^T \mathcal{R}_\varepsilon^*\big(-\mathrm{D}\mathcal{E}_\varepsilon(\mathfrak{r}_\varepsilon u)+\zeta\big)\,\mathrm{d}t = \int_0^T \Bigg\{ \frac{c_0}{2}\sum_{\iota\in\{-,+\}}\int_\Sigma q_B\big(\nabla_y(c_0^{-1}u^\iota-\zeta^\iota)\big)\,\mathrm{d}y + \frac{c_0}{2}\int_\Sigma \mathrm{harm}_{I_1^0}(a)\,[\![c_0^{-1}u-\zeta]\!]^2\,\mathrm{d}y\Bigg\}\,\mathrm{d}t = \int_0^T \mathcal{R}_{\mathrm{eff}}^*\big(-\mathrm{D}\mathcal{E}_0(u)+\zeta\big)\,\mathrm{d}t$$

It remains to show

$$\int_0^T \mathcal{R}_\varepsilon(\mathfrak{r}_\varepsilon\dot u)\,\mathrm{d}t \to \int_0^T \mathcal{R}_{\mathrm{eff}}(\dot u)\,\mathrm{d}t.$$

We use the quadratic structure $2\mathcal{R}_\varepsilon(\mathfrak{r}_\varepsilon\dot u)=\langle\mathbb{G}_\varepsilon\mathfrak{r}_\varepsilon\dot u,\mathfrak{r}_\varepsilon\dot u\rangle=\langle\mathfrak{r}_\varepsilon^*\mathbb{G}_\varepsilon\mathfrak{r}_\varepsilon\dot u,\dot u\rangle$ and show boundedness of $\mathfrak{D}_\varepsilon^{\mathrm{prim}}(\mathfrak{r}_\varepsilon u)$. We rewrite

$$\langle\mathbb{G}_\varepsilon\mathfrak{r}_\varepsilon\dot u,\mathfrak{r}_\varepsilon\dot u\rangle = \sup_{\substack{\xi\in\mathrm{X}_{\mathrm{av}}^*\\ \xi\neq0}}\frac{\langle\xi,\mathfrak{r}_\varepsilon\dot u\rangle^2}{\langle\xi,\mathbb{K}_\varepsilon\xi\rangle} \le \sup_{\varepsilon,\xi}\frac{\langle\mathfrak{r}_\varepsilon^*\xi,\mathbb{K}_0\mathfrak{r}_\varepsilon^*\xi\rangle}{\langle\xi,\mathbb{K}_\varepsilon\xi\rangle}\sup_{\substack{\xi\in\mathrm{X}_{\mathrm{av}}^*\\ \xi\neq0}}\frac{\langle\xi,\mathfrak{r}_\varepsilon\dot u\rangle^2}{\langle\mathfrak{r}_\varepsilon^*\xi,\mathbb{K}_0\mathfrak{r}_\varepsilon^*\xi\rangle}\le C\langle\mathbb{G}_0\dot u,\dot u\rangle$$

where $C:=\sup_{\varepsilon,\xi}\frac{\langle\mathfrak{r}_\varepsilon^*\xi,\mathbb{K}_0\mathfrak{r}_\varepsilon^*\xi\rangle}{\langle\xi,\mathbb{K}_\varepsilon\xi\rangle}$. We need to show, that $C<\infty$. With $\mathbb{K}_0$ given by the kinetic relation (4.13).

We compute

$$(\mathfrak{r}_\varepsilon^*\xi)^\pm = \int_{I^\pm}\xi\,\mathrm{d}z + \varepsilon^\delta\int_{I^0}p_a^\pm(z)\xi\,\mathrm{d}z$$

where $p_a^\pm$ from (4.14). Thus

$$\|\nabla(\mathfrak{r}_\varepsilon^*\xi)^+\|_{\mathrm{L}^2(\Sigma)}+\|\nabla(\mathfrak{r}_\varepsilon^*\xi)^-\|_{\mathrm{L}^2(\Sigma)} \le \|\nabla_\Sigma\xi\|_{\mathrm{L}^2(\Omega_1\setminus\Omega_1^0)}+2\varepsilon^\delta\|\nabla_\Sigma\xi\|_{\mathrm{L}^2(\Omega_1^0)}\le 2\|\sqrt{\mathbf{m}_\varepsilon}\nabla_\Sigma\xi\|_{\mathrm{L}^2(\Omega_1)}$$

where we used $|p_a^\pm|\le 1$. Obviously,

$$\|[\![\mathfrak{r}_\varepsilon^*\xi]\!]\|_{\mathrm{L}^2(\Sigma)}\le\|\textstyle\int_{I^+}\xi\,\mathrm{d}z-\int_{I^-}\xi\,\mathrm{d}z\|_{\mathrm{L}^2(\Sigma)}+\varepsilon^\delta\|\int_{I^0}(p_a^+-p_a^-)\xi\,\mathrm{d}z\|_{\mathrm{L}^2(\Sigma)}.$$

We easily estimate

$$\|\textstyle\int_{I^+}\xi\,\mathrm{d}z-\int_{I^-}\xi\,\mathrm{d}z\|_{\mathrm{L}^2(\Sigma)}\le\sqrt{3}\|\partial_z\xi\|_{\mathrm{L}^2(\Omega_1)}.$$

Using that $|p_a^+-p_a^-|\le 1$ and the Poincaré-Wirtinger estimate (cf. Lemma A.1) we obtain

$$\varepsilon^\delta\|\textstyle\int_{I^0}(p_a^+-p_a^-)\xi\,\mathrm{d}z\|_{\mathrm{L}^2(\Sigma)}\le c(\|\sqrt{\mathbf{m}_\varepsilon}\nabla_y\xi\|_{\mathrm{L}^2(\Omega_1)}+\|\partial_z\xi\|_{\mathrm{L}^2(\Omega_1)}).$$

Thus $\sup_{\varepsilon,\xi}\frac{\langle\mathfrak{r}_\varepsilon^*\xi,\mathbb{K}_0\mathfrak{r}_\varepsilon^*\xi\rangle}{\langle\xi,\mathbb{K}_\varepsilon\xi\rangle}<\infty$. In particular,

$$\sup_\varepsilon\int_0^T\mathcal{R}_\varepsilon(\mathfrak{r}_\varepsilon\dot u)\,\mathrm{d}t<\infty \quad\text{and}\quad \sup_\varepsilon\int_0^T\langle\mathfrak{r}_\varepsilon^*\mathbb{G}_\varepsilon\mathfrak{r}_\varepsilon\dot u,\mathbb{K}_0\mathfrak{r}_\varepsilon^*\mathbb{G}_\varepsilon\mathfrak{r}_\varepsilon\dot u\rangle\,\mathrm{d}t<\infty.$$

Let ξ_ε given by the kinetic relation (4.8) for $\mathfrak{r}_\varepsilon \dot{u}$. Using also that $\mathfrak{r}_\varepsilon^* \xi_\varepsilon$ is bounded in $\mathbb{X}^*$ and applying Lemma 4.6 gives that $\mathfrak{r}_\varepsilon^* \xi_\varepsilon \rightharpoonup \xi$ in $\mathbb{X}^*$ with $\mathbb{K}_0 \xi = \dot{u}$. Using the identities

$$\mathcal{R}_\varepsilon(\mathfrak{r}_\varepsilon \dot{u}) = \frac{1}{2}\langle \xi_\varepsilon, \mathfrak{r}_\varepsilon \dot{u}\rangle = \frac{1}{2}\langle \mathfrak{r}_\varepsilon^* \xi_\varepsilon, \dot{u}\rangle \quad \text{and} \quad \mathcal{R}_{\text{eff}}(\dot{u}) = \frac{1}{2}\langle \xi, \dot{u}\rangle$$

we conclude

$$\lim \mathfrak{D}_\varepsilon^\zeta(\mathfrak{r}_\varepsilon u) = \mathfrak{D}_{\text{eff}}^\zeta(u).$$

□

4.1.4 Convergence of the gradient flows

Collecting the results of section 4.1 we find the following theorem.

Theorem 4.9. *We have EDP-convergence with tilting of the gradient system FF* $(\mathrm{X}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ *to the effective gradient system* $(\mathbb{X}, \mathcal{E}_0, \mathcal{R}_{\text{eff}})$.

Proof. In the sequel we consider solutions u_ε to the gradient flow induced by the gradient system 4.7 and show that the limit of $\mathbb{L}^* u_\varepsilon$ satisfies the EDB for $(\mathbb{X}, \mathcal{E}_0, \mathcal{R}_{\text{eff}})$. We even consider the case when the limit of the initial conditions $\hat{u}_0 := \lim u_\varepsilon(0)$ is not flat, i.e., $\lim \mathbb{L}^* u_\varepsilon(0) \neq \hat{u}_0$ but the energy is bounded, i.e., $\mathcal{E}_\varepsilon\big(u_\varepsilon(0)\big) < C$. We deduce a priori almost everywhere convergence of the energies $\mathcal{E}_\varepsilon\big(u_\varepsilon(t)\big) \to \mathcal{E}_0\big(u_0(t)\big)$. This can be deduced from the a priori bounds (4.9) and the Aubin-Lions-Lemma [Sim87, Cor. 4]. In Section 4.2 the arguments and techniques for such result are shown in more detail. Thus we can pass to the limit in the EDB with an (almost) arbitrary 'initial' time $t_1 > 0$ and obtain

$$\begin{aligned} 0 = \liminf \mathcal{E}_\varepsilon\big(u_\varepsilon(T)\big) - \mathcal{E}_\varepsilon\big(u_\varepsilon(t_1)\big) + \mathfrak{D}_\varepsilon(u_\varepsilon, [t_1, T]) \\ \geq \mathcal{E}_0\big(u(T)\big) - \mathcal{E}_0\big(u(t_1)\big) + \mathfrak{D}_{\text{eff}}(u, [t_1, T]) \end{aligned}$$

In particular, u satisfies the differential inclusion

$$\text{for almost all } t \in (0, T) : -\mathrm{D}\mathcal{E}_0\big(u(t)\big) \in \partial \mathcal{R}_{\text{eff}}(\dot{u}).$$

However, we gained additionally that $[0, T] \ni t \mapsto \mathcal{E}_0\big(u(t)\big)$ is monotonously decreasing. Thus u is continuous in L^2 since the energy is equivalent to the norm-square. This is exploited in a refined limit passage

$$\begin{aligned} \mathcal{E}_0(\hat{u}_0) = \liminf \mathcal{E}_\varepsilon\big(u_\varepsilon(T)\big) + \mathfrak{D}_\varepsilon(u_\varepsilon, [t_1, T]) + \mathcal{E}_\varepsilon\big(u_\varepsilon(0)\big) - \mathcal{E}_\varepsilon\big(u_\varepsilon(t_1)\big) \\ = \mathcal{E}_0\big(u(T)\big) + \mathfrak{D}_{\text{eff}}(u, [t_1, T]) + \mathcal{E}_0(\hat{u}_0) - \mathcal{E}_0\big(u(t_1)\big) \end{aligned}$$

Passing to the limit $t_1 \to 0$ we obtain

$$\mathcal{E}_0\big(u(T)\big) + \mathfrak{D}_{\text{eff}}(u, [0, T]) = \mathcal{E}_0\big(u(0)\big).$$

□

Note that Lemma 4.3 yields $\mathbb{L}^*\hat{u}_0 = u(0)$. This shows that we have EDP-convergence with tilting, even when the initial datum is not well-prepared. Moreover, for this doubly quadratic structure $(\mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ we obtain the doubly quadratic limiting structure $(\mathcal{E}_0, \mathcal{R}_{\text{eff}})$ as expected due to [Bra06, Prop 2.13] since $\mathfrak{D}_\varepsilon$ is quadratic.

4.2 The Boltzmann-Wasserstein setting

This section is concerned with the limit passage for the gradient system (4.2b). The main result here is that the effective dissipation potential is no longer quadratic but involves $R^*_{\text{jump}}(u^\pm, [\![\xi]\!]) = 4\sqrt{u^+u^-}\big(\cosh([\![\xi]\!]/2) - 1\big)$ to describe the jump process. The main issue concerning the limit passage is the density u_ε of μ_ε with respect to π_ε since μ_ε vanishes on the middle layer but u_ε does not.

We transform the gradient system given in (4.2b) by the push-forward measure $(\Phi_\varepsilon)_\#\boldsymbol{\mu} = \mu$ with Φ_ε from (4.5). The forces are then given by $\xi = \boldsymbol{\xi} \circ \Phi_\varepsilon^{-1}$. Hence, we obtain

$$
\begin{aligned}
\mathbf{X} &= \mathcal{P}(\overline{\Omega}_1),\\
\mathcal{E}_\varepsilon(\mu) &= \begin{cases} \displaystyle\int_{\Omega_1} \mathrm{E}_1\left(\frac{\mathrm{d}\mu}{\mathrm{d}\pi_\varepsilon}\right)\mathrm{d}\pi_\varepsilon & \text{if } \mu \ll \pi_\varepsilon,\\ \infty & \text{else,}\end{cases}\\
\mathcal{R}^*_\varepsilon(\mu,\xi) &= \int_{\Omega_1} \frac{1}{2}\nabla'\xi\cdot B_\varepsilon(x)\nabla'\xi\,\mathrm{d}\mu + \int_{\Omega_1^0}\frac{a}{2}|\partial_d\xi|^2 u\,\mathrm{d}\Pi_\varepsilon + \int_{\Omega_1\setminus\Omega_1^0}\frac{a}{2\varepsilon^2}|\partial_d\xi|^2\,\mathrm{d}\mu,
\end{aligned}
$$

with $\nabla' = (\partial_j)_{j=1,\cdots,d-1}$, $\mathrm{d}\pi_\varepsilon = Z_\varepsilon\mathbf{m}_\varepsilon\,\mathrm{d}x$,

$$
\mathbf{m}_\varepsilon(z) = \begin{cases} 1 & \text{if } z\in I^\pm\\ \varepsilon^\delta & \text{if } z \in I^0\end{cases}
$$

and $\mathrm{d}\Pi_\varepsilon = Z_\varepsilon\,\mathrm{d}x$ is the normalized Lebesgue measure. The normalization constant is given by $Z_\varepsilon^{-1} = (2+\varepsilon^\delta)|\Sigma|$. The transformation of the energie $\hat{\mathcal{E}}^{(2)}_\varepsilon$ and the dissipation potential $\hat{\mathcal{R}}^{(2)}_\varepsilon$ follows from

$$
\begin{aligned}
\nabla'\xi = \mathrm{I}_{d-1}(\nabla'\boldsymbol{\xi})\circ\Phi_\varepsilon^{-1}, \partial_d\xi = \varepsilon^{-1}(\partial_d\boldsymbol{\xi})\circ\Phi_\varepsilon^{-1} \text{ on } \Omega_1\setminus\Omega_1^0\\
\text{and } \partial_d\xi = \varepsilon^{-1-\delta}(\partial_d\boldsymbol{\xi})\circ\Phi_\varepsilon^{-1} \text{ on } \Omega_1^0.
\end{aligned}
$$

Moreover, $\mathbf{a}_{\varepsilon|_{\Omega_1^0}}\circ\Phi^{-1} = \varepsilon^{2+\delta}a$ and $\mathrm{d}\mu_{|_{\Omega_1^0}} = \varepsilon^\delta u\,\mathrm{d}x$. The primal dissipation potential reads

$$
\mathcal{R}_\varepsilon(\mu,\dot\mu) = \int_{\Omega_1}\frac{1}{2}v'\cdot B_\varepsilon(x)v'\,\mathrm{d}\mu + \int_{\Omega_1^0}\frac{a}{2}|v_d|^2u\,\mathrm{d}\Pi_\varepsilon + \int_{\Omega_1\setminus\Omega_1^0}\frac{a}{2\varepsilon^2}|v_d|^2\,\mathrm{d}\mu,
$$

where $(v', v_d)\in \mathrm{H}(\mathrm{d}\mu,\Omega_1)$ with H introduced in Section 2.1 satisfies

$$
\forall\,\xi\in\mathrm{W}^{1,\infty}(\Omega_1): \langle\xi,\dot\mu\rangle = \int_{\Omega_1} v'\cdot B_\varepsilon(x)\nabla'\xi\,\mathrm{d}\mu + \int_{\Omega_1^0} av_d\partial_d\xi\;u\,\mathrm{d}\Pi_\varepsilon + \int_{\Omega_1\setminus\Omega_1^0}\frac{a}{\varepsilon^2}v_d\partial_d\xi\,\mathrm{d}\mu. \tag{4.15}
$$

The dual dissipation potential evaluated at the driving force, which is also called slope term, reads

$$\begin{aligned}\mathcal{R}_\varepsilon\big(\mu,-\mathrm{D}\mathcal{E}_\varepsilon(\mu)\big)=&\int_{\Omega_1}\frac{1}{2}\nabla' u\cdot B_\varepsilon(x)\nabla' u\,\frac{1}{u}\,\mathrm{d}\pi_\varepsilon+\int_{\Omega_1^0}\frac{a}{2}\frac{|\partial_d u|^2}{u}\,\mathrm{d}\Pi_\varepsilon\\&+\int_{\Omega_1\setminus\Omega_1^0}\frac{a}{2\varepsilon^2}\frac{|\partial_d u|^2}{u}\,\mathrm{d}\pi_\varepsilon.\end{aligned}\tag{4.16}$$

4.2.1 Compactness

In the sequel, we prove a priori bounds for the densities u_ε and obtain a weak compactness result for $u_{\varepsilon|\Omega_1^\pm}$ on the top and bottom layer and a weak* compactness result for $u_{\varepsilon|\Omega_1^0}$ on the middle layer. For the velocity $\dot\mu_\varepsilon$ we obtain a pointwise BV-bound. The first part is devoted to a priori estimates derived from the bound on $\mathfrak{D}_\varepsilon^{\mathrm{dual}}$ and the second part is devoted to derive a priori bounds from the bound on $\mathfrak{D}_\varepsilon^{\mathrm{prim}}$ and $\mathcal{E}_\varepsilon$ with

$$\mathfrak{D}_\varepsilon^{\mathrm{dual}}(\mu,[0,T])=\int_0^T\mathcal{R}_\varepsilon^*\big(\mu,-\mathrm{D}\mathcal{E}_\varepsilon(\mu)\big)\,\mathrm{d}t\quad\text{and}\quad\mathfrak{D}_\varepsilon^{\mathrm{prim}}(\mu,[0,T])=\int_0^T\mathcal{R}_\varepsilon(\mu,\dot\mu).$$

As a consequence, we derive strong convergence for the densities u_ε and almost everywhere in time convergence of the energies. Note that solutions with bounded energie at the initial datum satisfy the natural bound

$$\sup_{\varepsilon>0}\sup_{t\in[0,T]}\mathcal{E}_\varepsilon(\mu_\varepsilon(t))<\infty\quad\text{and}\quad\sup_{\varepsilon>0}\mathfrak{D}_\varepsilon(\mu_\varepsilon,[0,T])<\infty.\tag{4.17}$$

Using estimates derived in Section 2.2 and the mass constraint $\mu_\varepsilon(\Omega_1)=1$ we obtain bounded mass of the density and equi-integrability of $\nabla u_\varepsilon|_{\Omega_1^+\cup\Omega_1^-}$.

Lemma 4.10. *Assume that the family of curves $t\mapsto\mu_\varepsilon(t)$ satisfies*

$$\sup_\varepsilon\mathfrak{D}_\varepsilon^{\mathrm{dual}}(\mu_\varepsilon,[0,T])<\infty.\tag{4.18}$$

Then, u_ε satisfies the estimates

$$\sup_{\varepsilon>0}\int_0^T\int_{\Omega_1^0}\big(|u_\varepsilon|+|\partial_d u_\varepsilon|\big)\,\mathrm{d}x\,\mathrm{d}t<\infty\quad\textit{and}\quad\sup_{\varepsilon>0}\int_0^T\int_{\Omega_1^\pm}\Big(|\nabla' u_\varepsilon|+\frac{|\partial_d u_\varepsilon|}{\varepsilon}\Big)\,\mathrm{d}x\,\mathrm{d}t<\infty.$$

If we assume additionally that

$$\sup_{\varepsilon>0,t\in[0,T]}\mathcal{E}_\varepsilon\big(\mu_\varepsilon(t)\big)<\infty,$$

then the family $\{\nabla u_\varepsilon|_{\Omega_1^+\cup\Omega_1^-}\}_{\varepsilon>0}\subset\mathrm{L}^1([0,T]\times(\Omega_1^+\cup\Omega_1^-);\mathbb{R}^d)$ is equi-integrable.

Proof. We use (2.2) which reads

$$\int_{\Omega_1^0} u_\varepsilon \,\mathrm{d}x \le \int_{\Omega_1^0} |\partial_d u_\varepsilon| \,\mathrm{d}x + \int_{\Omega_1^\pm} \big(|\partial_d u_\varepsilon| + u_\varepsilon\big) \,\mathrm{d}x.$$

Moreover, using $|\partial_d u_\varepsilon| = |\partial_d \log u_\varepsilon| u_\varepsilon$ and Young's estimate $2|\partial_d u_\varepsilon| \le |\partial_d \log u_\varepsilon|^2 u_\varepsilon + u_\varepsilon$ we obtain

$$\int_{\Omega_1^0} u_\varepsilon \,\mathrm{d}x \le \int_{\Omega_1^0} \frac{1}{2}\big(|\partial_d \log u_\varepsilon|^2 u_\varepsilon + u_\varepsilon\big) \,\mathrm{d}x + \int_{\Omega_1^\pm} \frac{1}{2}\big(|\partial_d \log u_\varepsilon|^2 u_\varepsilon + 3u_\varepsilon\big) \,\mathrm{d}x.$$

By the bound on the slope term (4.16) we obtain

$$\sup_{\varepsilon>0} \int_0^T \int_{\Omega_1^0} u_\varepsilon \,\mathrm{d}x \,\mathrm{d}t < \infty.$$

Thus, applying Young's estimate again yields

$$\sup_{\varepsilon>0} \int_0^T \int_{\Omega_1^0} |\partial_d u_\varepsilon| \,\mathrm{d}x \,\mathrm{d}t < \infty \quad \text{and} \quad \sup_{\varepsilon>0} \int_0^T \int_{\Omega_1^\pm} \Big(\frac{|\partial_d u_\varepsilon|}{\varepsilon}\Big) \,\mathrm{d}x \,\mathrm{d}t < \infty.$$

Finally, the second estimate in (2.4) for $w_\varepsilon = \nabla' u_\varepsilon$ and $v_\varepsilon = u_\varepsilon$ gives for any measurable $A \in \mathcal{B}([0,T] \times \Omega_1^\pm)$ that

$$\int_A |\nabla' u_\varepsilon| \,\mathrm{d}x \,\mathrm{d}t \le \bigg(\int_0^T \int_{\Omega_1^\pm} |\nabla' \log u_\varepsilon|^2 u_\varepsilon \,\mathrm{d}x \,\mathrm{d}t\bigg)^{1/2} \Big(\int_A u_\varepsilon \,\mathrm{d}x \,\mathrm{d}t\Big)^{1/2}.$$

Note that by the bound on the energy, we have that u_ε is equi-integrable. Hence, $\nabla' u_\varepsilon$ is equi-integrable since the slope term is bounded. □

Next, we prove pointwise in time compactness for any curve satisfying the natural bounds (4.17). However, due to the behavior of μ_ε on the middle layer Ω_1^0, we cannot expect uniform estimates in the space of absolutely continuous curves in the 2-Wasserstein space as e.g. in [AMP+12].

Due to Lemma 4.10, limits u_0 will be constant in the vertical direction in the upper and lower layers Ω_1^+ and Ω_1^-, respectively. Hence, we define the reduction map $R : \Omega_1 \to \Omega_1^0$ via

$$R(x_1,x_2,x_3) = \begin{cases} (x_1,x_2,z^+) & \text{for } x_3 \in [z^+, z^+ + 1], \\ (x_1,x_2,x_3) & \text{for } x_3 \in \,]z^-, z^+[\\ (x_1,x_2,z^-) & \text{for } x_3 \in [z^- - 1, z^-]. \end{cases} \tag{4.19}$$

By considering the push-forward of measures $\mu \in \mathcal{P}(\overline{\Omega}_1)$ under the map R we arrive at reduced measures $\eta = R_\# \mu \in \mathcal{P}(\overline{\Omega}_1^0)$ for which we will consider the following decomposition

$$\eta := R_\# \nu = \eta^+ \otimes \delta_{z^+} + \eta^0 + \eta^- \otimes \delta_{z^-}, \tag{4.20}$$

where $\eta^\pm \in \mathrm{Meas}(\Sigma)$ with $\eta^\pm(A) = \mu(A \times I_1^\pm)$ for a Borel set $A \subset \Sigma$ and $\eta^0 = \mu|_{\Omega_1^0}$.

Lemma 4.11. *Let μ_ε be such that*

$$\sup_\varepsilon \mathfrak{D}_\varepsilon^{\mathrm{prim}}(\mu_\varepsilon,[0,T]) < \infty.$$

Then the total variation of the reduced measures $t \mapsto \eta_\varepsilon(t) = R_\# \mu_\varepsilon(t)$ with respect to the 1-Wasserstein metric on $\mathcal{P}(\Omega_1^0)$ is uniformly bounded, i.e.

$$\sup\Big\{ \sum_{j=1}^N \mathcal{W}_1\big(\eta_\varepsilon(t_j),\eta_\varepsilon(t_{j-1})\big) \,\Big|\, 0=t_0<t_1<\ldots<t_N=T, \varepsilon>0 \Big\} < \infty.$$

Moreover, we have that $\int_{I_1^\pm} \dot u_\varepsilon \,\mathrm{d}z$ is bounded in $\mathrm{L}^1\big(0,T;\mathrm{W}_0^{1,\infty}(\Sigma)^*\big)$ and $\varepsilon^\delta \dot u_{\varepsilon|_{\Omega_1^0}}$ is bounded in $\mathrm{L}^1\big(0,T;\mathrm{W}_0^{1,\infty}(\Omega_1^0)^*\big)$.

Proof. We exploit the well known dual formulation of the 1-Wasserstein distance in terms of 1-Lipschitz continuous function [AGS05], i.e. for probability measures $\eta_1,\, \eta_2 \in \mathcal{P}(\Sigma \times [z^-,z^+])$ we have that

$$\mathcal{W}_1(\eta_1,\eta_2) = \sup\Big\{ \int_{\Omega_1^0} \varphi(x)\,\mathrm{d}\eta_1(x) - \int_{\Omega_1^0} \varphi(x)\,\mathrm{d}\eta_2(x) \,\Big|\, \varphi \in \mathrm{C}^{\mathrm{Lip}}(\Omega_1^0),\, \mathrm{Lip}(\varphi)\le 1 \Big\}.$$

For a given $\varphi \in \mathrm{C}^{\mathrm{Lip}}(\Omega_1^0)$ with $\mathrm{Lip}(\varphi) \le 1$ let us denote by $\bar\varphi$ its extension to Ω_1, i.e. $\bar\varphi := \varphi \circ R$. Then, by the property of the push-forward and Young's estimate, we arrive at

$$\begin{aligned}\int_{\Omega_1^0} \varphi(x)\,\mathrm{d}(\eta_\varepsilon(t_j) - \eta_\varepsilon(t_{j-1})) &= \int_{\Omega_1} \bar\varphi(x)\,\mathrm{d}(\mu_\varepsilon(t_j) - \mu_\varepsilon(t_{j-1})) \\ &= \int_{t_{j-1}}^{t_j} \langle \bar\varphi, \dot\mu_\varepsilon\rangle\,\mathrm{d}t \le \int_{t_{j-1}}^{t_j} \Big\{ \mathcal{R}_\varepsilon^*(\mu_\varepsilon,\bar\varphi) + \mathcal{R}_\varepsilon(\mu_\varepsilon,\dot\mu_\varepsilon)\Big\}\,\mathrm{d}t.\end{aligned} \tag{4.21}$$

Using $|\nabla\bar\varphi| \le 1$ almost everywhere we estimate by the ellipticity constant Λ from (4.3)

$$\mathcal{R}_\varepsilon^*(\mu_\varepsilon;\bar\varphi) \le Z_\varepsilon \Lambda \int_{\Omega_1} u_\varepsilon \,\mathrm{d}x.$$

Hence, by Lemma 4.10 we obtain that the right hand side of

$$\sum_{j=1}^N \mathcal{W}_1(\eta_\varepsilon(t_j),\eta_\varepsilon(t_{j-1})) \le \int_0^T \Big\{ \mathcal{R}_\varepsilon(\mu_\varepsilon,\dot\mu_\varepsilon) + Z_\varepsilon\Lambda \int_{\Omega_1} u_\varepsilon\,\mathrm{d}x\Big\}\,\mathrm{d}t,$$

is uniformly bounded. Testing (4.21) with $\varphi \in \mathrm{L}^\infty\big(0,T;\mathrm{W}_0^{1,\infty}(\Sigma)\big)$ we obtain that $\int_{I_1^\pm} \dot u_\varepsilon\,\mathrm{d}z$ is bounded in $\mathrm{L}^1\big(0,T;\mathrm{W}_0^{1,\infty}(\Sigma)^*\big)$. Here we used [Roc71] giving that for $\dot u_\varepsilon \in \mathrm{L}^1(0,T;X^*)$ we have that the dual norm of $\mathrm{L}^\infty(0,T;X)$ is the norm of $\mathrm{L}^1(0,T;X^*)$. Similarly, we find $\varepsilon^\delta \dot u_{\varepsilon|_{\Omega_1^0}}$ is bounded in $\mathrm{L}^1\big(0,T;\mathrm{W}_0^{1,\infty}(\Omega_1^0)^*\big)$. □

In particular, using the bound on the energies Helly's selection principle [DM09] gives (up to a subsequence) a pointwise limit $\eta_\varepsilon(t) \rightharpoonup^* \eta_0(t)$ in $\mathcal{P}(\overline{\Omega}_1^0)$ for every $t \in [0,T]$. Moreover, we find a (non-relabeled) subsequences and limits such that

$$\mu_\varepsilon \rightharpoonup^* \mu_0 \text{ in } \mathrm{Meas}([0,T]\times\overline{\Omega}_1) \quad \text{and} \quad Z_\varepsilon u_\varepsilon \rightharpoonup^* N_0 \text{ in } \mathrm{Meas}([0,T]\times\overline{\Omega}_1). \tag{4.22}$$

with the additional convergences $u_\varepsilon \rightharpoonup u_0$ in $\mathrm{L}^1\big([0,T] \times \mathrm{W}^{1,1}(\Omega_1^\pm)\big)$ satisfying $\partial_d u_\varepsilon \to 0$ in $\mathrm{L}^1\big([0,T] \times \Omega_1^\pm\big)$ and $\partial_d u_\varepsilon \rightharpoonup^* H_0$ in $\mathrm{Meas}([0,T]\times\overline{\Omega}_1^0)$.

Using the Aubin-Lion-Lemma [Sim87, Cor. 4] we deduce strong convergence of μ_ε in $\mathrm{L}^1\big(0,T;\mathrm{L}^p(\Omega_1)\big)$ for some $p > 1$ and hence, almost everywhere convergence of the energies, i.e.,

$$\mathcal{E}_\varepsilon\big(\mu_\varepsilon(t)\big) \to \mathcal{E}_0\big(\eta(t)\big) \quad \text{for almost all } t \in [0,T].$$

Lemma 4.12. *Let the family of curves $t \mapsto \mu_\varepsilon(t)$ satisfies*

$$\sup_\varepsilon \mathfrak{D}_\varepsilon(\mu_\varepsilon,[0,T]) < \infty. \tag{4.23}$$

Then, there exists a subsequence such that $\int_{I_1^\pm} u_\varepsilon|_{\Omega_1^\pm}\,\mathrm{d}z \to u$ in $\mathrm{L}^1\big(0,T;\mathrm{L}^{p_1}(\Sigma)\big)$ for $\frac{d-1}{d-2} > p_1 > 1$ and for any $\alpha > 0$ there exists $p_1 \geq p_2 > 1$ such that $\varepsilon^\alpha u_\varepsilon|_{\Omega_1^0} \to 0$ in $\mathrm{L}^1\big(0,T;\mathrm{L}^{p_2}(\Omega_1^0)\big)$.

Proof. Lemma 4.11 states that $\int_{I_1^\pm} \dot{u}_\varepsilon\,\mathrm{d}z$ is bounded in $\mathrm{L}^1\big(0,T;\mathrm{W}_0^{1,\infty}(\Sigma)^*\big)$ and Lemma 4.10 states that $u_\varepsilon|_{\Omega_1^\pm}$ is bounded in $\mathrm{L}^1\big([0,T] \times \mathrm{W}^{1,1}(\Omega_1^\pm)\big)$. We recall that $\Sigma \subset \mathbb{R}^{d-1}$. For $p_1 < \frac{d-1}{d-2}$ if $d > 2$ and $p_1 \in [1,\infty)$ arbitrary if $d = 2$ we have the compact embedding of $\mathrm{W}^{1,1}(\Sigma)$ into $\mathrm{L}^{p_1}(\Sigma)$. Thus by [Sim87, Cor. 4] it follows strong compactness of $\int_{I_1^\pm} u_\varepsilon|_{\Omega_1^\pm}\,\mathrm{d}z$ in $\mathrm{L}^1\big(0,T;\mathrm{L}^{p_1}(\Sigma)\big)$. Again, Lemma 4.11 gives boundedness of $\varepsilon^\delta \dot{u}_\varepsilon|_{\Omega_1^0}$ in $\mathrm{L}^1\big(0,T;\mathrm{W}_0^{1,\infty}(\Omega_1^0)^*\big)$. By the method in the proof of Lemma 4.10 we find $\varepsilon^{\delta+\gamma} u_\varepsilon|_{\Omega_1^0}$ is bounded in $\mathrm{L}^1\big([0,T] \times \mathrm{W}^{1,1}(\Omega_1^0)\big)$, where γ comes from the scaling $B_\varepsilon|_{\Omega_1^0} = \varepsilon^\gamma B$. Hence, we find $\varepsilon^{\delta+\gamma} u_\varepsilon|_{\Omega_1^\pm} \to 0$ in $\mathrm{L}^1\big(0,T;\mathrm{L}^{p_1}(\Omega_1^0)\big)$. By the bound on $u_\varepsilon|_{\Omega_1^0}$ in $\mathrm{L}^1\big((0,T) \times \Omega_1^0\big)$ we estimate

$$\int_0^T \|u_\varepsilon\|_{\mathrm{L}^{p_2}}\,\mathrm{d}t \leq \left(\int_0^T \|u_\varepsilon\|_{\mathrm{L}^1}\,\mathrm{d}t\right)^{\theta/p_2} \left(\int_0^T \|u_\varepsilon\|_{\mathrm{L}^{p_1}}\,\mathrm{d}t\right)^{\frac{p_1(p_2-1)}{p_2(p_1-1)}},$$

where θ satisfies $(p_1 - p_2) = \theta(p_1 - 1)$. For any $\alpha > 0$ and for p_2 close enough to 1 we find $\varepsilon^\alpha u_\varepsilon|_{\Omega_1^0} \to 0$ in $\mathrm{L}^1\big(0,T;\mathrm{L}^{p_2}(\Omega_1^0)\big)$. □

We define the limit energy

$$\mathcal{E}_0(\eta) = \begin{cases} \int_{\Omega_1} \mathrm{E}_1\left(\frac{\mathrm{d}\eta}{\mathrm{d}\vartheta}\right)\mathrm{d}\vartheta & \text{if } \eta \ll \vartheta, \\ \infty & \text{else,} \end{cases}$$

where $\vartheta = Z_0\big(\mathrm{d}y\otimes\delta_{z^+} + \mathrm{d}y\otimes\delta_{z^-}\big)$. In particular, for $\eta = Z_0\big(u^+\,\mathrm{d}y\otimes\delta_{z^+} + u^-\,\mathrm{d}y\otimes\delta_{z^-}\big)$ the energy reads

$$\mathcal{E}_0(\eta) = \int_\Sigma \big(\mathrm{E}_1(u^+) + \mathrm{E}_1(u^-)\big) Z_0\,\mathrm{d}y.$$

Lemma 4.13. *Let μ_ε satisfy the natural bound (4.17) such that $R_\#\mu_\varepsilon(t) \rightharpoonup^* \eta(t)$ for every $t\in[0,T]$. Then we have indeed up to a subsequence $\mathcal{E}_\varepsilon\big(\mu_\varepsilon(t)\big) \to \mathcal{E}_0\big(\eta(t)\big)$ for almost all $t\in[0,T]$.*

Proof. Note that by virtue of Lemma 4.12 we have $\varepsilon^\alpha u_\varepsilon(t) \to 0$ in $\mathrm{L}^{p_2}(\Omega_1^0)$ for some $p_2 > 1$ and any $\alpha > 0$. Hence, $\int_{\Omega_1^0} \mathrm{E}_1\big(u_\varepsilon(t)\big)\varepsilon^\delta\,\mathrm{d}x \to 0$. Jensen's estimate with respect to the probability measure $\mathrm{d}z$ and $(\int_{I_1^\pm} u_\varepsilon\,\mathrm{d}z)^{-1} u_\varepsilon\,\mathrm{d}z$ (using concavity of $\log$) yields

$$\mathrm{E}_1\Big(\textstyle\int_{I_1^\pm} u_\varepsilon\,\mathrm{d}z\Big) \le \int_{I_1^\pm} \mathrm{E}_1(u_\varepsilon)\,\mathrm{d}z \le \int_{I_1^\pm} u_\varepsilon\,\mathrm{d}z \log\Big(\int_{I_1^\pm} u_\varepsilon^2\,\mathrm{d}z\Big) - \mathrm{E}_1\Big(\int_{I_1^\pm} u_\varepsilon\,\mathrm{d}z\Big).$$

We already know, that $\mathrm{E}_1\Big(\int_{I_1^\pm} u_\varepsilon(t)\,\mathrm{d}z\Big) \to \mathrm{E}_1\big(u^\pm(t)\big)$ in $\mathrm{L}^1(\Sigma)$. It remains to show that

$$\int_{I_1^\pm} u_\varepsilon(t)\,\mathrm{d}z \log\Big(\int_{I_1^\pm} u_\varepsilon^2(t)\,\mathrm{d}z\Big) \to 2\mathrm{E}_1\big(u^\pm(t)\big) \quad \text{in } \mathrm{L}^1(\Sigma). \tag{4.24}$$

Since $\partial_d u_\varepsilon \to 0$ in $\mathrm{L}^1\big((0,T)\times\Omega_1^\pm\big)$ we deduce $u_\varepsilon|_{\Omega_1^\pm} \to u^\pm$ a.e. in $(0,T)\times\Omega_1^\pm$, where $u^\pm \in \mathrm{L}^1\big(0,T;\mathrm{L}^{p_1}(\Sigma)\big)$. Hence, we obtain in (4.24) convergence a.e. in Σ. Applying the Vitali convergence theorem we conclude via the estimate

$$\int_{I_1^\pm} u^2\,\mathrm{d}z \le c\bigg(\int_{I_1^\pm} |\partial_d u| + |u|\,\mathrm{d}z\bigg)^2$$

and the fact that $\int_{I_1^\pm} u_\varepsilon(t)\,\mathrm{d}z$ converges in $\mathrm{L}^{p_1}(\Sigma)$ and that $(\log x)^p \le x - c_p$ for $x \ge 1$ and any $p \ge 0$. □

In order to pass to a Γ-$\liminf$ for the total dissipation functional we need to exploit additional regularity properties given by the bound on the dissipation leading to more regularity of both, the weak* limit of $u_\varepsilon|_{\Omega_1^0}$ and the distributional derivative $\dot\eta$.

4.2.2 The Γ-liminf estimate of $\mathfrak{D}_\varepsilon$

First, we note that u is constant in vertical direction in the top and bottom layer, i.e., $(\partial_d u)|_{\Omega_1^\pm} \equiv 0$. We denote the restrictions to the top and bottom layer by $u^\pm := u|_{\Omega_1^\pm} \in \mathrm{L}^1(\Sigma)$. This gives rise to an effective coefficient matrix on the top and bottom layers that is given by

$$\overline{B}_\pm := \int_{I_1^\pm} B\,\mathrm{d}z. \tag{4.25}$$

In the sequel, we prove liminf-estimates for the primal and dual dissipation separately. The obtained lower bounds depend on the limit N_0 of $Z_\varepsilon u_\varepsilon$ in both, the primal and dual part. In order to relate N_0 to the limit $u^\pm$ we use the bound on the dual dissipation and decompose the measure N_0 via the disintegration theorem ([AGS05, Thm 5.3.1], Section 2.2) into $\mathrm{d}N_0 = \mathrm{d}N_{y,t}\,\mathrm{d}N$, where $N : \mathcal{B}([0,T]\times\Sigma) \ni B \mapsto N_0(B\times\overline{I}_1)$. By assigning traces to the fiber measure $N_{y,t} \in \mathrm{Meas}(\overline{I}_1)$ on the middle layer we solve a minimization problem and calculate the minimum value explicitly.

Note that for measures N_0 with a density with respect to the Lebesgue measure, i.e., $\mathrm{d}N_0 = u\,\mathrm{d}(x,t)$ we have that

$$\mathrm{d}N = \left(\int_{I_1} u(y,z,t)\,\mathrm{d}z\right)\mathrm{d}y\,\mathrm{d}t \quad \text{and} \quad \mathrm{d}N_{y,t} = \frac{u(y,z,t)}{\int_{I_1} u\,\mathrm{d}z}\,\mathrm{d}z.$$

In the following we consider the average along the fibers of the middle layer $N^0 : A \mapsto N_0(A\times\overline{I}_1^0)$ for the limit passage in the middle layer. However, we also use the average along the fibers of the whole domain to relate the traces $u^\pm$ to the measure restricted to the middle layer $N_{0|_{[0,T]\times\overline{\Omega}_1^0}}$.

Lemma 4.14. *Let $\mu_\varepsilon \rightharpoonup^* \mu$ in $\mathrm{Meas}([0,T]\times\overline{\Omega}_1)$ such that for all $t\in[0,T]$ we have $R_\#\mu_\varepsilon(t) \rightharpoonup^* \eta(t)$ in $\mathrm{Meas}(\overline{\Sigma})$, $Z_\varepsilon u_\varepsilon \rightharpoonup^* N_0$ in $\mathrm{Meas}([0,T]\times\overline{\Omega}_1)$ and*

$$\sup_\varepsilon \left\{ \mathfrak{D}_\varepsilon^{\mathrm{dual}}(\mu_\varepsilon,[0,T]) + \sup_{t\in[0,T]} \mathcal{E}_\varepsilon\big(\mu_\varepsilon(t)\big) \right\} < \infty.$$

We denote $N : A \mapsto N_0(A\times I_1)$ and $N^0 : A \mapsto N_0(A\times\overline{I}_1^0)$ with $A \in \mathcal{B}([0,T]\times\overline{\Sigma})$. Then $\mathrm{d}N_0 = f_{y,t}\,\mathrm{d}z\,\mathrm{d}N$ and $\mathrm{d}N_{0|_{\overline{\Omega}_1^0}} = g_{y,t}\,\mathrm{d}z\,\mathrm{d}N^0$ with $f_{y,t} \in \mathrm{W}^{1,1}(I_1)$ N-a.e. and $g_{y,t} \in \mathrm{W}^{1,1}(I_1^0)$ N^0-a.e. satisfying the relation $f_{y,t|_{\overline{I}^0}} = g_{y,t}\frac{\mathrm{d}N^0}{\mathrm{d}N}$.

Moreover, there exists non-negative $u^\pm \in \mathrm{L}^1\big((0,T)\times\Sigma\big)$ such that $\mathrm{d}N_{0|_{\Omega_1^\pm}} = u^\pm\,\mathrm{d}x\,\mathrm{d}t$ satisfying

$$\liminf \int_0^T \int_{\Omega_1\setminus\Omega_1^0} \nabla' u_\varepsilon B \nabla' u_\varepsilon \frac{1}{u_\varepsilon}\,\mathrm{d}x\,\mathrm{d}t \geq \int_0^T \sum_{\iota\in\{-,+\}} \int_\Sigma \nabla' u^\iota \overline{B}_\iota \nabla' u^\iota \frac{1}{u^\iota}\,\mathrm{d}y\,\mathrm{d}t. \tag{4.26}$$

On the middle layer we obtain the following liminf-estimate.

$$\liminf \int_0^T \int_{\overline{\Omega}_1^0} a \left|\frac{\partial_d u_\varepsilon}{u_\varepsilon}\right|^2 u_\varepsilon\,\mathrm{d}\Pi_\varepsilon\,\mathrm{d}t \geq \int_{[0,T]\times\overline{\Sigma}} \int_{I^0} a \left|\frac{\partial_d g_{y,t}}{g_{y,t}}\right|^2 g_{y,t}\,\mathrm{d}z\,\mathrm{d}N^0(y,t). \tag{4.27}$$

Proof. First, we establish weak*-l.s.c. of the slope-term, that is we show for $Z_\varepsilon u_\varepsilon \rightharpoonup^* N_0$ $\mathrm{Meas}([0,T]\times\overline{\Omega}_1^0)$ and $Z_\varepsilon \partial_d u_\varepsilon \rightharpoonup^* H_0$ in $\mathrm{Meas}([0,T]\times\overline{\Omega}_1^0)$ it holds

$$\liminf \int_0^T \int_{\overline{\Omega}_1^0} \left|\frac{\partial_d u_\varepsilon}{u_\varepsilon}\right|^2 u_\varepsilon\,\mathrm{d}\Pi_\varepsilon\,\mathrm{d}t \geq \int_0^T \int_{\overline{\Omega}_1^0} \left|\frac{\mathrm{d}H_0}{\mathrm{d}N_0}\right|^2 \mathrm{d}N_0(x,t).$$

We restrict ourselves to the domain $\overline{\Omega}_1^0$ since we already know that $\mathrm{d}N_{0|_{\overline{\Omega}_1^\pm}} = Z_0 u^\pm \,\mathrm{d}y\,\mathrm{d}t$. We denote $\overline{\Omega}_{1T}^0 = [0,T] \times \overline{\Omega}_1^0$ and use Young's estimate to obtain

$$\int_{\overline{\Omega}_{1T}^0} \frac{|\partial_d u_\varepsilon|^2}{u_\varepsilon} \,\mathrm{d}x\,\mathrm{d}t = \int_{\overline{\Omega}_{1T}^0} u_\varepsilon \left(\left| \frac{\partial_d u_\varepsilon}{u_\varepsilon} \right|^2 + F^2 - F^2 \right) \mathrm{d}x\,\mathrm{d}t \geq \int_{\overline{\Omega}_{1T}^0} \big(2F\partial_d u_\varepsilon - F^2 u_\varepsilon \big) \,\mathrm{d}x\,\mathrm{d}t$$

Hence, for all $F \in \mathrm{C}^0(\overline{\Omega}_{1T}^0)$ we have

$$\liminf \int_{\overline{\Omega}_{1T}^0} \frac{|\partial_d u_\varepsilon|^2}{u_\varepsilon} \,\mathrm{d}\Pi_\varepsilon \,\mathrm{d}t \geq \int_{\overline{\Omega}_{1T}^0} 2F \,\mathrm{d}H_0 - \int_{\overline{\Omega}_{1T}^0} F^2 \,\mathrm{d}N_0.$$

If there exists a measurable $A \in \mathcal{B}(\overline{\Omega}_{1T}^0)$ such that $|H_0|(A) > 0$ but $N_0(A) = 0$ then there exists a sequence $F_n \in \mathrm{C}^0(\overline{\Omega}_{1T}^0)$ concentrating on A such that

$$\int_{\overline{\Omega}_1^0} 2F_n \,\mathrm{d}H_0 - \int_{\overline{\Omega}_1^0} F_n^2 \,\mathrm{d}N_0 \to \infty.$$

Since the slope term is bounded we conclude that $N_0 \gg H_0$. By approximation of $\frac{\mathrm{d}H_0}{\mathrm{d}N_0}$ by a sequence $F_n \in \mathrm{C}^0(\overline{\Omega}_{1T}^0)$ we obtain

$$\int_{\overline{\Omega}_1^0} 2F \,\mathrm{d}H_0 - \int_{\overline{\Omega}_1^0} F^2 \,\mathrm{d}N_0 = \int_{\overline{\Omega}_1^0} \Big(2F_n \frac{\mathrm{d}H_0}{\mathrm{d}N_0} - F_n^2 \Big) \,\mathrm{d}N_0 \to \int_{\overline{\Omega}_1^0} \left| \frac{\mathrm{d}H_0}{\mathrm{d}N_0} \right|^2 \mathrm{d}N_0.$$

Similarly, we estimate

$$\int_{\overline{\Omega}_{1T}^0} \frac{|\nabla' u_\varepsilon|_B^2}{u_\varepsilon} \,\mathrm{d}x\,\mathrm{d}t \geq \int_{\overline{\Omega}_{1T}^0} \big(2\langle F, \nabla' u_\varepsilon \rangle_B - |F|_B^2 u_\varepsilon \big) \,\mathrm{d}x\,\mathrm{d}t,$$

where $\langle x_1, x_2 \rangle_B = x_1 \cdot B x_2$. Observing, that $u_{|_{\Omega_1^\pm}}$ does not depend on z, the estimate (4.26) follows.

The following reasoning holds for both $N_{y,t}$ and $N_{y,t}^0$ but we elaborate only on the latter. Moreover, we argue only N^0-a.e. (resp. N-a.e.). Using $H_0 \ll N_0$, we easily see that $\mathrm{d}H_{0|_{\overline{\Omega}_1^0}} = \mathrm{d}N_{y,t}^0{}' \mathrm{d}N^0$ with $\frac{\mathrm{d}N_{y,t}^0{}'}{\mathrm{d}N_{y,t}^0} = \frac{\mathrm{d}H_0}{\mathrm{d}N_0}$. By weak*-convergence of u_ε we obtain that $N_{y,t}^0{}'$ and $N_{y,t}^0$ satisfy the following differential relation: $\forall\, \varphi \in \mathrm{C}^0(\overline{\Omega}_{1T}^0)$ with $\partial_d \varphi \in \mathrm{C}^0(\overline{\Omega}_{1T}^0)$ and $\varphi(t,y,z^\pm) \equiv 0$

$$\int_{[0,T]\times\Sigma} \int_{I_1^0} \partial_z \varphi \,\mathrm{d}N_{y,t}^0 \,\mathrm{d}N^0 = -\int_{[0,T]\times\Sigma} \int_{I_1^0} \varphi \,\mathrm{d}N_{y,t}^0{}' \,\mathrm{d}N^0.$$

Using [AFP00, Thm 3.30] we may represent $N_{y,t}^0{}'$ as a derivative of a BV-function $g_{y,t}$ on I_1^0 and conclude $\mathrm{d}N_{y,t}^0 = g_{y,t}\,\mathrm{d}z$. By the relation $N_{y,t}^0{}' \ll N_{y,t}^0 \ll \mathrm{d}z$ we conclude even $g_{y,t} \in \mathrm{W}^{1,1}(I_1^0)$ and $\mathrm{d}N_{y,t}^0{}' = \partial_d g_{y,t}\,\mathrm{d}z$. This proves (4.27). The $\mathrm{W}^{1,1}(I)$-function for $N_{y,t}$ is denoted by $f_{y,t}$, i.e., $\mathrm{d}N_{y,t} = f_{y,t}\,\mathrm{d}z$.

Denoting $N^{\pm} : A \mapsto N_0(A \times I_1^{\pm})$ and $N_{y,t}^{\pm}$ given by the disintegration theorem we verify the decomposition

$$\mathrm{d}N_{y,t} = \mathbb{1}_{I_1^+} \frac{\mathrm{d}N^+}{\mathrm{d}N} \mathrm{d}N_{y,t}^+ + \mathbb{1}_{I_1^0} \frac{\mathrm{d}N^0}{\mathrm{d}N} \mathrm{d}N_{y,t}^0 + \mathbb{1}_{I_1^-} \frac{\mathrm{d}N^-}{\mathrm{d}N} \mathrm{d}N_{y,t}^-$$

by integration $\varphi \in \mathrm{C}^0([0,T] \times \overline{\Omega}_1)$ with respect to the measure N_0, i.e.,

$$\int_{\overline{\Omega}_{1T}} \varphi \,\mathrm{d}N_0 = \int_{[0,T]\times\Sigma} \sum_{\iota\in\{-,0,+\}} \int_{I_1^\iota} \varphi \,\mathrm{d}N_{y,t}^\iota \,\mathrm{d}N^\iota = \int_{[0,T]\times\Sigma} \int_{I_1} \varphi \,\mathrm{d}N_{y,t} \,\mathrm{d}N.$$

Note that $N^\iota \ll N$ since for all $A \in \mathcal{B}([0,T] \times \overline{\Sigma})$ we have $N^\iota(A) \leq N(A)$ for $\iota \in \{-,0,+\}$. □

As a corollary we obtain the liminf estimate also when a tilt $\zeta \in \mathrm{W}^{1,\infty}(\Omega_1^0)$ with boundary values $\zeta^{\pm} := \zeta(\cdot, z^{\pm}) \in \mathrm{W}^{1,\infty}(\Sigma)$ is present.

Corollary 4.15. *Let the assumptions of Lemma 4.14 hold true. Let $\bar{\zeta} = \zeta \circ R$ be the extension of ζ. Then*

$$\liminf \int_0^T \int_{\overline{\Omega}_1^0} a \left| \frac{\partial_d u_\varepsilon}{u_\varepsilon} - \partial_d \bar{\zeta} \right|^2 u_\varepsilon \,\mathrm{d}\Pi_\varepsilon \,\mathrm{d}t \geq \int_{[0,T]\times\overline{\Sigma}} \int_{I_1^0} a \left| \frac{\partial_d g_{y,t}}{g_{y,t}} - \partial_d \bar{\zeta} \right|^2 g_{y,t} \,\mathrm{d}z \,\mathrm{d}N^0(y,t)$$

and

$$\liminf_{\varepsilon\downarrow 0} \int_0^T \int_{\Omega_1\setminus\Omega_1^0} \nabla'(\log u_\varepsilon - \bar{\zeta}) B \nabla'(\log u_\varepsilon - \bar{\zeta}) u_\varepsilon \,\mathrm{d}x \,\mathrm{d}t$$
$$\geq \int_0^T \sum_{\iota\in\{-,+\}} \int_\Sigma \nabla'(\log u^\iota - \zeta^\iota) \overline{B}_\iota \nabla'(\log u^\iota - \zeta^\iota) u^\iota \,\mathrm{d}y \,\mathrm{d}t.$$

Proof. We observe $|\partial_{x_j}(\log u - \bar{\zeta})|^2 u = |\partial_{x_j} \log u|^2 u - 2\partial_{x_j}\bar{\zeta}\partial u + |\partial_{x_j}\bar{\zeta}|^2 u$. Hence, by weak* convergence of $\partial_{x_j} u_\varepsilon$ and u_ε and Lemma 4.14 the result follows. □

We obtain traces for $g_{y,t}$ in terms of $u^{\pm}$. Moreover, we need below the fact that $Z_0\sqrt{u^- u^+}\,\mathrm{d}y\,\mathrm{d}t = \sqrt{g_{y,t}(z^-) g_{y,t}(z^+)}\,\mathrm{d}N^0$.

Corollary 4.16. *Let the assumptions of Lemma 4.14 hold true. Then it holds*

$$Z_0 u^{\pm} \,\mathrm{d}y \,\mathrm{d}t = g_{y,t}(z^{\pm}) \,\mathrm{d}N^0 \quad \textit{and} \quad u^{\pm} g_{y,t}(z^{\mp}) = u^{\mp} g_{y,t}(z^{\pm})$$
$$\textit{and} \quad Z_0 \sqrt{u^- u^+}\,\mathrm{d}y \,\mathrm{d}t = \sqrt{g_{y,t}(z^-) g_{y,t}(z^+)}\,\mathrm{d}N^0.$$

Proof. Note that on $\Omega_1 \setminus \Omega_1^0$ we have $Z_0 u^{\pm} \,\mathrm{d}x \,\mathrm{d}t = f_{y,t}(z) \,\mathrm{d}z \,\mathrm{d}N$. Thus it follows that $Z_0 u^{\pm} \,\mathrm{d}y \,\mathrm{d}t = f_{y,t}(z^{\pm}) \,\mathrm{d}N$. Lemma 4.14 gives the relation $f_{y,t}(z^{\pm}) \,\mathrm{d}N = g_{y,t}(z^{\pm}) \,\mathrm{d}N^0$. Moreover, with $\mathrm{d}N^{\pm} := g_{y,t}(z^{\pm}) \,\mathrm{d}N^0$ we conclude the identity

$$\sqrt{g_{y,t}(z^-) g_{y,t}(z^+)}\,\mathrm{d}N^0 = \sqrt{\frac{\mathrm{d}N^-}{\mathrm{d}(N^+ + N^-)} \frac{\mathrm{d}N^+}{\mathrm{d}(N^+ + N^-)}}\,\mathrm{d}(N^+ + N^-)$$
$$= \sqrt{\frac{u^-}{u^+ + u^-} \frac{u^+}{u^+ + u^-}}\, Z_0(u^+ + u^-) \,\mathrm{d}y \,\mathrm{d}t = Z_0 \sqrt{u^- u^+}\,\mathrm{d}y \,\mathrm{d}t.$$

The relation $u^{\pm} g_{y,t}(z^{\mp}) = u^{\mp} g_{y,t}(z^{\pm})$ follows from

$$Z_0 u^{\pm} g_{y,t}(z^{\mp})\,\mathrm{d}y\,\mathrm{d}t = g_{y,t}(z^{\pm}) g_{y,t}(z^{\mp})\,\mathrm{d}N^0 = Z_0 u^{\mp} g_{y,t}(z^{\pm})\,\mathrm{d}y\,\mathrm{d}t.$$

□

For notational convenience we write $g_{y,t}^{\pm} := g_{y,t}(z^{\pm})$. Thus we have a Γ-$\liminf$ estimate for the dual part of the dissipation depending on N_0 which is connected to $u^{\pm}$ via the disintegration theorem. For the primal part we use the Wasserstein theory of [AGS05] introduced in Section 2.2.

Lemma 4.17. *Let* $\mu_\varepsilon \rightharpoonup^* \mu$ *in* $\mathrm{Meas}([0,T] \times \overline{\Omega}_1)$ *such that for all* $t \in [0,T]$ *we have* $R_{\#}\mu_\varepsilon(t) \rightharpoonup^* \eta(t)$ *in* $\mathrm{Meas}(\overline{\Sigma})$*,* $Z_\varepsilon u_\varepsilon \rightharpoonup^* N_0$ *in* $\mathrm{Meas}([0,T] \times \overline{\Omega}_1)$ *and*

$$\sup_\varepsilon \left\{ \mathfrak{D}_\varepsilon^{\mathrm{dual}}(\mu_\varepsilon, [0,T]) + \sup_{t \in [0,T]} \mathcal{E}_\varepsilon\big(\mu_\varepsilon(t)\big) \right\} < \infty.$$

Then we have

$$\liminf_{\varepsilon \downarrow 0} \int_0^T \mathcal{R}_\varepsilon(\mu_\varepsilon, \dot{\mu}_\varepsilon)\,\mathrm{d}t \geq \int_0^T \sum_{\iota \in \{-,+\}} \int_\Sigma w_\iota' \cdot \overline{B}_\iota w_\iota' Z_0 u^\iota \,\mathrm{d}y\,\mathrm{d}t + \int_{[0,T]\times\overline{\Omega}_1^0} \frac{|\tilde{\kappa}|^2}{a g_{y,t}}\,\mathrm{d}z\,\mathrm{d}N^0$$

where $(w^{\pm}, \tilde{\kappa})$ *satisfy the limiting continuity equation*

$$\int_0^T \int_\Sigma \Big(\sum_{\iota \in \{-,+\}} \big(\nabla' \varphi^\iota \overline{B}_\iota w^\iota - \dot{\varphi}^\iota\big) Z_0 u^\iota \Big)\,\mathrm{d}y\,\mathrm{d}t + \int_0^T \int_\Sigma \tilde{\kappa} [\![\varphi]\!]\,\mathrm{d}N^0 = 0.$$

for all $\varphi = (\varphi^-, \varphi^+) \in \mathrm{C}_c^\infty([0,T] \times \Sigma) \times \mathrm{C}_c^\infty([0,T] \times \Sigma)$

Proof. By Lemma 2.2 we obtain a suitable notion of weak convergence and a limit $v \in \mathrm{L}^2\big(N_0, [0,T] \times \overline{\Omega}_1\big)$ such that $v_\varepsilon \rightharpoonup v$ satisfying the liminf-estimate

$$\liminf_{\varepsilon \downarrow 0} \int_0^T \int_{\Omega_1} v_\varepsilon' \cdot B v_\varepsilon' \,\mathrm{d}\mu_\varepsilon\,\mathrm{d}t \geq \int_0^T \int_{\Omega_1 \setminus \Omega_1^0} v' \cdot B v' \,\mathrm{d}\mu\,\mathrm{d}t$$

and

$$\liminf_{\varepsilon \downarrow 0} \int_0^T \int_{\Omega_1^0} a |v_{d,\varepsilon}|^2 u_\varepsilon \,\mathrm{d}\Lambda_\varepsilon\,\mathrm{d}t \geq \int_{[0,T]\times\overline{\Omega}_1^0} a |v_d|^2 \,\mathrm{d}N_0.$$

Passing to the limit of the continuity equation (4.15) for $\varphi \in \mathrm{C}_c^\infty([0,T] \times \Omega_1)$ such that $(\partial_d \varphi)_{|_{\Omega_1^{\pm}}} \equiv 0$ we obtain

$$\int_0^T \int_{\Omega_1 \setminus \Omega_1^0} \big(\nabla' \varphi B v' - \dot{\varphi}\big)\,\mathrm{d}\mu\,\mathrm{d}t + \int_{[0,T]\times\overline{\Omega}_1^0} a \partial_d \varphi v_d \,\mathrm{d}N_0 = 0. \tag{4.28}$$

In particular, using $\mathrm{d}N_0 = g_{y,t}\,\mathrm{d}z\,\mathrm{d}N^0$ and testing with φ such that $\mathrm{supp}(\varphi) \subset [0,T] \times \overline{\Omega}_1^0$ we obtain that $\partial_z(a v_d g_{y,t}) = 0$, i.e., $\tilde{\kappa} := a v_d g_{y,t}$ is constant with respect to z since $\mu(\Omega_1^0) = 0$. We recall that $\mu_{|_{\Omega_1^\pm}} = Z_0 u^\pm \,\mathrm{d}x$. Note that since $(\partial_d \varphi)_{|_{\Omega_1^\pm}} = (\partial_d u)_{|_{\Omega_1^\pm}} \equiv 0$ we obtain that for any solution (v', v_d) of (4.28) we have that (w', v_d) such that

$$\int_{I_1^\pm} Bv' \,\mathrm{d}z = \int_{I_1^\pm} Bw' \,\mathrm{d}z$$

is a solution to (4.28), too. Hence, $(w', \tilde{\kappa})$ defined via

$$w'_{|_{\Omega_1^\pm}} = w^\pm := \overline{B}_\pm^{-1} \int_{I_1^\pm} Bv' \,\mathrm{d}z \quad \text{with } \overline{B} \text{ from (4.25)}$$

is also a solution to

$$\int_0^T \int_\Sigma \Big(\sum_{\iota \in \{-,+\}} \big(\nabla \varphi^\iota \overline{B}_\iota w^\iota - \dot{\varphi}^\iota\big) Z_0 u^\iota \Big) \,\mathrm{d}y \,\mathrm{d}t + \int_{[0,T]\times\overline{\Sigma}} \tilde{\kappa} [\![\varphi]\!] \,\mathrm{d}N^0 = 0.$$

A simple computation yields that

$$\int_{\Omega_1^\pm} w' \cdot Bw' \; u^\pm \,\mathrm{d}x \geq \int_\Sigma w^\pm \overline{B}_\pm w^\pm u^\pm \,\mathrm{d}y$$

for all w' satisfying the constraint $\overline{B}_\pm w^\pm = \int_{I_1^\pm} Bw' \,\mathrm{d}z$. This finishes the proof. $\square$

On the middle layer we obtained a contribution from both, the primal part $\liminf_{\varepsilon \downarrow 0} \mathfrak{D}^{\mathrm{prim}}(\mu_\varepsilon)$ and the tilted dual part $\liminf_{\varepsilon\downarrow 0} \mathfrak{D}^{\mathrm{dual}\zeta}(\mu_\varepsilon)$, that depends on $\mathrm{d}N_0 = g_{y,t}(z) \,\mathrm{d}z \,\mathrm{d}N^0$ and reads

$$\mathfrak{F}(g, \tilde{\kappa}; \zeta) := \frac{1}{2} \int_{[0,T]\times\overline{\Sigma}} \int_{I_1^0} a|\partial_d \log g_{y,t} - \partial_d \zeta|^2 g_{y,t} + \frac{\tilde{\kappa}^2}{a g_{y,t}} \,\mathrm{d}z \,\mathrm{d}N^0.$$

The minimization problem $\min \mathfrak{F}(\cdot, \tilde{\kappa}; \zeta)$ is solved in [LMPR17, Proposition A.2] and it's value can be explicitly calculated.

Lemma 4.18 ([LMPR17, Proposition A.2]). *Denoting* $\kappa = \frac{\tilde{\kappa}}{\mathrm{harm}_{I_1^0}(a\mathrm{e}^\zeta)\sqrt{g_{y,t}^+ g_{y,t}^- \mathrm{e}^{-(\zeta^+ + \zeta^-)}}}$ *with* $\mathrm{harm}_{I_1^0}\left(a\mathrm{e}^\zeta\right)$ *defined in* (4.4) *we have that*

$$\min_g \mathfrak{F}(\cdot, \tilde{\kappa}; \zeta) \;=\; \mathrm{harm}_{I_1^0}\left(a\mathrm{e}^\zeta\right) \sqrt{g_{y,t}^+ g_{y,t}^- \mathrm{e}^{-(\zeta^+ + \zeta^-)}} \big(\mathscr{C}(\kappa) \;+\; \mathscr{C}^*([\![\log(g_{y,t}) \;-\; \zeta]\!])\big),$$

where the minimum is taken over all $g_{y,t}$ *with traces* $g_{y,t}(z^\pm) = g_{y,t}^\pm$ *and* $\mathscr{C}^*$ *is given by*

$$\mathscr{C}^*(\xi) = 4\big(\cosh(\xi/2) - 1\big).$$

Corollary 4.16 states that $\sqrt{g^+_{y,t} g^-_{y,t}}\,\mathrm{d}N^0 = Z_0\sqrt{u^+u^-}\,\mathrm{d}y\,\mathrm{d}t$ and $g^+_{y,t}/g^-_{y,t} = u^+/u^-$. Hence, we arrive at

$$\begin{aligned}\liminf_{\varepsilon\downarrow 0} \mathfrak{D}^\zeta_\varepsilon(\mu_\varepsilon;[0,T]) \geq \\ \int_0^T \Bigg(\int_\Sigma \Big(\sum_{\iota\in\{-,+\}} \big(\nabla(\log u^\iota - \zeta^\iota)\overline{B}_\iota \nabla(\log u^\iota - \zeta^\iota) + w'_\iota \cdot \overline{B}_\iota w'_\iota\big) Z_0 u^\iota\Big)\,\mathrm{d}y \\ + \int_\Sigma \operatorname{harm}\big(a\mathrm{e}^\zeta\big)\sqrt{u^+u^-\mathrm{e}^{-(\zeta^++\zeta^-)}}\big(\mathscr{C}(\kappa) + \mathscr{C}^*([\![\log u - \zeta]\!])\big)\,\mathrm{d}y\Bigg)\,\mathrm{d}t. \end{aligned} \tag{4.29}$$

with (w,κ) satisfying the continuity equation

$$\int_0^T \int_\Sigma \Big(\sum_{\iota\in\{-,+\}} \big(\nabla\varphi^\iota \overline{B}_\iota w^\iota - \dot\varphi^\iota\big)u^\iota\Big) + \mathsf{a}(a,\zeta)\kappa[\![\varphi]\!]\sqrt{u^-u^+}\,\mathrm{d}y\,\mathrm{d}t = 0 \tag{4.30}$$

for all $\varphi = (\varphi^-,\varphi^+) \in \mathrm{C}^\infty_c([0,T]\times\Sigma)\times\mathrm{C}^\infty_c([0,T]\times\Sigma)$ with the effective coefficient

$$\mathsf{a}(a,\zeta) := \operatorname{harm}\big(a\mathrm{e}^\zeta\big)\sqrt{\mathrm{e}^{-(\zeta^++\zeta^-)}}. \tag{4.31}$$

The remaining part of this subsection is devoted to weak differentiability of the limiting curve η and to define the effective dissipation potential. We introduce the space Y which is defined as the closure of $\mathrm{L}^\infty\big(0,T;\mathrm{C}^1(\Sigma)\times\mathrm{C}^1(\Sigma)\big)$ with respect to the norm

$$\|\Theta\|_Y = \|\nabla\xi^+\|_{\mathrm{L}^2_{u^+}} + \|\nabla\xi^-\|_{\mathrm{L}^2_{u^-}} + \|[\![\xi]\!]\|_{\mathrm{L}^{\mathscr{C}^*}_{\sqrt{u^+u^-}}},$$

where $\Theta = (\nabla\xi^+,\nabla\xi^-,[\![\xi]\!])$ and for $0\leq v\in\mathrm{L}^1\big((0,T)\times\Sigma\big)$ we have

$$\|\Xi\|^2_{\mathrm{L}^2_v} = \int_0^T\int_\Sigma |\Xi|^2 v\,\mathrm{d}y\,\mathrm{d}t \quad \text{and} \quad \|\xi\|_{\mathrm{L}^{\mathscr{C}^*}_v} = \inf\Big\{k>0 : \int_0^T\int_\Sigma \mathscr{C}^*(\xi/k)v\,\mathrm{d}y\,\mathrm{d}t \leq 1\Big\}.$$

For an introduction to Orlicz spaces $\mathrm{L}^{\mathscr{C}^*}_v$ we refer to [RZ91]. We just need the fact that there holds a Hölder estimate

$$\int_{(0,T)\times\Sigma} \kappa[\![\xi]\!]v\,\mathrm{d}y\,\mathrm{d}t \leq 2\|\kappa\|_{\mathrm{L}^{\mathscr{C}}_v}\|[\![\xi]\!]\|_{\mathrm{L}^{\mathscr{C}^*}_v}$$

and that $\big(\mathrm{L}^{\mathscr{C}}_v\big)^* = \mathrm{L}^{\mathscr{C}^*}_v$ since $\mathscr{C}(2x)\leq C\mathscr{C}(x)$. By (4.29) we have that $w^\pm \in \mathrm{L}^2_{u^+}$ and $\kappa\in\mathrm{L}^{\mathscr{C}}_{\sqrt{u^+u^-}}$. In particular, using Hölder's estimate we obtain

$$\int_0^T\int_\Sigma u^+\dot\varphi^+ + u^-\dot\varphi^-\,\mathrm{d}y\,\mathrm{d}t \leq C(\|w^+\|_{\mathrm{L}^2_{u^+}} + \|w^-\|_{\mathrm{L}^2_{u^-}} + \|\kappa\|_{\mathrm{L}^{\mathscr{C}}_{\sqrt{u^+u^-}}})\|\Theta\|_Y,$$

where $\Theta = (\nabla\varphi^+,\nabla\varphi^-,[\![\varphi]\!])$ and C depends only on Λ from (4.3) and $\|\zeta\|_{\mathrm{L}^\infty(\Omega^0_1)}$. Since

$$\eta = Z_0\big(u^+\,\mathrm{d}y\otimes\delta_{z^+} + u^-\,\mathrm{d}y\otimes\delta_{z^-}\big)$$

we have $\dot\eta \in Y^*$. We choose the decomposition $\dot\eta = \dot\eta_w + \dot\eta_{\zeta,\kappa}$ with

$$\langle \dot\eta_w, \Theta\rangle = \int_\Sigma \Big(\sum_{\iota\in\{-,+\}} \nabla\varphi^\iota \cdot \overline{B}_\iota w^\iota Z_0 u^\iota \Big) \,\mathrm{d}y$$

and

$$\langle \dot\eta_{\zeta,\kappa}, \Theta\rangle = \int_\Sigma Z_0 \mathsf{a}(a,\zeta)\sqrt{u^+u^-}\kappa[\![\varphi]\!]\,\mathrm{d}y$$

where $\Theta = (\nabla\varphi^-, \nabla\varphi^+, [\![\varphi]\!])$ and $\mathsf{a}(a,\zeta)$ given in (4.31). We define

$$\mathcal{R}_{\mathrm{diff}}(\eta,\dot\eta) = \begin{cases} \int_\Sigma w\overline{B}w\,\mathrm{d}\eta & \text{if } \dot\eta = \dot\eta_w, \\ \infty & \text{else} \end{cases}$$

and

$$\mathcal{R}^\zeta_{\mathrm{jump}}(\eta,\dot\eta) = \begin{cases} \int_\Sigma \operatorname{harm}\left(a\mathrm{e}^\zeta\right)\sqrt{\mathrm{e}^{-(\zeta^++\zeta^-)}}\mathscr{C}(\kappa)\,\mathrm{d}\mathbb{GM}(\eta) & \text{if } \dot\eta = \dot\eta_{\zeta,\kappa}, \\ \infty & \text{else.} \end{cases} \tag{4.32}$$

The geometric mean of two positive measures $\eta = (\eta^-,\eta^+)$ is defined as

$$\mathbb{GM}(\eta) := \sqrt{\frac{\mathrm{d}\eta^+}{\mathrm{d}(\eta^++\eta^-)}\frac{\mathrm{d}\eta^-}{\mathrm{d}(\eta^++\eta^-)}}\,\mathrm{d}(\eta^++\eta^-).$$

If both $\mathrm{d}\eta^\pm = Z_0 u^\pm\,\mathrm{d}x$ then $\mathbb{GM}(\eta) = Z_0\sqrt{u^+u^-}\,\mathrm{d}x$. The state space is then

$$\mathscr{X} := \left\{\eta = (\eta^-,\eta^+) \in \mathrm{Meas}(\overline{\Sigma}) \times \mathrm{Meas}(\overline{\Sigma}) | \eta^\pm \geq 0, (\eta^++\eta^-) \in \mathcal{P}(\overline{\Sigma})\right\}.$$

The effective dissipation potential is defined via the inf-convolution of $\mathcal{R}_{\mathrm{diff}}$ and $\mathcal{R}^\zeta_{\mathrm{jump}}$, i.e.,

$$\mathcal{R}^\zeta_{\mathrm{eff}}(\eta,\dot\eta) = \inf\left\{\mathcal{R}_{\mathrm{diff}}(\eta,\dot\eta_w) + \mathcal{R}^\zeta_{\mathrm{jump}}(\eta,\dot\eta_{\zeta,\kappa}) | \dot\eta = \dot\eta_w + \dot\eta_{\zeta,\kappa}\right\}$$

It is well known that the dual dissipation potential is then given by $\mathcal{R}^*_{\mathrm{diff}} + \mathcal{R}^{\zeta\,*}_{\mathrm{jump}}$ (see e.g. [AB86, Roc66]).

We recall that for a dissipation potential $\mathcal{R}$ and energy $\mathcal{E}$ the tilted total dissipation functional is given by

$$\mathfrak{D}^\zeta(\mu;[0,T]) = \int_0^T \mathcal{R}(\mu,\dot\mu) + \mathcal{R}^*\big(\mu, -\mathrm{D}\mathcal{E}(\mu)+\zeta\big)\,\mathrm{d}t.$$

Combining Lemma 4.14, Lemma 4.17 and Lemma 4.18 we obtain the following liminf-estimate for the tilted total dissipation functional.

Theorem 4.19. *Let* $\mu_\varepsilon \rightharpoonup^* \mu$ *in* $\mathrm{Meas}([0,T]\times\overline{\Omega}_1)$ *such that* $R_\# \mu_\varepsilon(t) \rightharpoonup^* \eta(t)$ *in* $\mathrm{Meas}([0,T]\times\overline{\Sigma})$ *pointwise in* $[0,T]$, $Z_\varepsilon u_\varepsilon \rightharpoonup^* N_0$ *in* $\mathrm{Meas}([0,T]\times\overline{\Omega}_1)$. *Moreover, let* (4.17) *hold. Then* $\mu_{|_{\Omega_1^\pm}} = Z_0 u^\pm\,\mathrm{d}x$ *and* $\mu(\Omega_1^0) = 0$ *and*

$$\liminf_{\varepsilon\downarrow 0} \mathfrak{D}^\zeta_\varepsilon(\mu_\varepsilon;[0,T]) \geq \mathfrak{D}^\zeta_{\mathrm{eff}}(\eta;[0,T]).$$

Note that $\mathcal{R}^{\zeta}_{\text{jump}}$ depends only on $\zeta_{|_{I_1^0}}$ and $\big(\zeta(z^+)+\zeta(z^-)\big)$. More precisely, the effective coeffiction $\mathsf{a}(a,\zeta)$ depends only on $\zeta_{|_{I_1^0}}$ and $\big(\zeta(z^+)+\zeta(z^-)\big)$. The effective force ξ enters the dual $\mathcal{R}^{\zeta*}_{\text{jump}}$ only in terms of $(\xi(z^+)-\xi(z^-)=[\![\xi]\!]$. Hence, $\mathcal{R}^{\zeta}_{\text{jump}}$ can be seen as independent of the effective force ζ. Moreover, if

$$\operatorname{harm}\left(a\mathrm{e}^{\zeta}\right)\sqrt{\mathrm{e}^{-(\zeta^{+}+\zeta^{-})}}=\operatorname{harm}\left(a\right)$$

then the effective dissipation potential is independent of ζ.

4.2.3 The Γ-limsup estimate of $\mathfrak{D}_{\varepsilon}$

This subsection is concerned with the construction of a recovery sequence for $\mathfrak{D}^{\zeta}_{\text{eff}}$ at $\eta_0\in\mathrm{L}^1\big(0,T;\mathcal{P}(\overline{\Omega}_1^0)\big)$ with $\mathrm{d}\eta_0=Z_0(u_0^+(y)\,\mathrm{d}y\otimes\delta_{z^+}+u_0^-(y)\,\mathrm{d}y\otimes\delta_{z^-})$. For this we use the representation

$$\mathcal{R}^{\zeta}_{\text{eff}}(\eta_0,\dot\eta_0)=\langle\varphi,\dot\eta_0\rangle-\mathcal{R}^{\zeta*}_{\text{eff}}(\eta_0,\dot\eta_0)$$

for $\xi_0\in\partial\mathcal{R}^{\zeta}_{\text{eff}}(\eta_0,\dot\eta_0)$, i.e., ξ_0 satisfies the continuity equation

$$\langle\varphi,\dot\eta_0\rangle=\int_{\Sigma}\Big\{\sum_{\iota\in\{-,+\}}\nabla\varphi^{\iota}\cdot\overline{B}_{\iota}\nabla\xi_0^{\iota}\;u_0^{+}\Big\}+\mathsf{a}(a,\zeta)\mathscr{C}^{*'}([\![\xi_0]\!])[\![\varphi]\!]\sqrt{u_0^+u_0^-}\,\mathrm{d}y \tag{4.33}$$

for all $\varphi=(\varphi^-,\varphi^+)\in\mathrm{C}^\infty(\Sigma)\times\mathrm{C}^\infty(\Sigma)$. In Theorem 4.21 below we give the precise statement followed by the rigorous proof. But first, we give characterizations for $\mathcal{R}^{*}_{\text{jump}}$ and main ideas of the rigorous proof.

We assume that the density $u_0^\pm$ satisfies the bound $0<\alpha\leq u_0^\pm$ for some $\alpha>0$. By the lower bound we embed the solution to the continuity equation $\xi_0\in\partial\mathcal{R}^{\zeta}_{\text{eff}}(\eta_0,\dot\eta_0)$ into the linear space $\mathrm{L}^2\big(0,T;\mathrm{H}^1(\Sigma)\times\mathrm{H}^1(\Sigma)\big)$ with $\int_\Sigma\xi_0^++\xi_0^-\,\mathrm{d}y=0$ such that the jump $[\![\xi_0]\!]$ is in $\mathrm{L}^{\mathscr{C}^*}\big((0,T)\times\Sigma\big)$. For fixed $u\in\mathrm{L}^1(\Omega_1)$ we equip the space

$$\mathrm{W}_{\varepsilon}(u)=\left\{\xi:\int_{\Omega_1}\xi\mathbf{m}_{\varepsilon}\,\mathrm{d}x=0\quad\text{and}\quad\int_{\Omega_1}|\nabla\xi|^2u\,\mathrm{d}x<\infty\right\} \tag{4.34a}$$

with the norm

$$\|\xi\|^2_{\mathrm{W}_\varepsilon(u)}=\int_{\Omega_1}|\nabla'\xi|^2\mathbf{m}_\varepsilon u\,\mathrm{d}x+\varepsilon^{-2}\int_{\Omega_1\backslash\Omega_1^0}|\partial_d\xi|^2u\,\mathrm{d}x+\int_{\Omega_1^0}|\partial_d\xi|^2u\,\mathrm{d}x.$$

We compare $\mathrm{W}_\varepsilon(u)$ to the space H^1_ε defined via

$$\mathrm{H}^1_{\varepsilon}=\left\{\xi:\int_{\Omega_1}\xi\mathbf{m}_{\varepsilon}\,\mathrm{d}x=0\quad\text{and}\quad\int_{\Omega_1}|\nabla\xi|^2\,\mathrm{d}x<\infty\right\} \tag{4.34b}$$

equipped with the norm

$$\|\xi\|^2_{\mathrm{H}^1_\varepsilon}=\int_{\Omega_1}|\nabla'\xi|^2\mathbf{m}_\varepsilon\,\mathrm{d}x+\int_{\Omega_1}|\partial_d\xi|^2\,\mathrm{d}x.$$

If we assume a lower bound $0 < \alpha \leq u$, then we deduce $\alpha\|\xi\|^2_{\mathrm{H}^1_\varepsilon} \leq \|\xi\|^2_{\mathrm{W}_\varepsilon(u)}$. By the Poincaré estimate (see Lemma A.1) we additionally have for some $c > 0$ independent of ε that

$$\alpha \int_{\Omega_1} \xi^2 \mathbf{m}_\varepsilon \,\mathrm{d}x \leq c\alpha\|\xi\|^2_{\mathrm{H}^1_\varepsilon} \leq c\|\xi\|^2_{\mathrm{W}_\varepsilon(u)}.$$

hence, we bound the dissipation.

Lemma 4.20. *If the recovery sequence $\mu_\varepsilon = u_\varepsilon \pi_\varepsilon$ also satisfies $\alpha \leq u_\varepsilon$ and*

$$\sup_{\varepsilon>0} \int_0^T \int_{\Omega_1} \dot{u}_\varepsilon^2 \mathbf{m}_\varepsilon \,\mathrm{d}x \,\mathrm{d}t < \infty,$$

then $\dot{\mu}_\varepsilon$ is bounded in $\mathrm{L}^2\big(0,T;\mathrm{W}_\varepsilon(u_\varepsilon)^*\big)$, i.e.,

$$\sup_{\varepsilon>0} \int_0^T \mathcal{R}_\varepsilon(\mu_\varepsilon, \dot{\mu}_\varepsilon)\,\mathrm{d}t = \sup_{\varepsilon>0} \mathfrak{D}^{\mathrm{prim}}_\varepsilon(\mu_\varepsilon;[0,T]) < \infty.$$

Thus we are able to pass to the limit using linear theory. Note that $\eta_0 \in \mathrm{L}^1\big(0,T;\mathcal{P}(\overline{\Omega}{}_1^0)\big)$ of the form $\mathrm{d}\eta_0 = Z_0(u_0^+(y)\,\mathrm{d}y \otimes \delta_{z^+} + u_0^-(y)\,\mathrm{d}y\otimes\delta_{z^-})$ can be embedded into $\mathrm{L}^1\big(0,T;\mathcal{P}(\overline{\Omega}_1)\big)$ via $\mu_0 = Z_0 u_0^+ \mathscr{L}_{|_{\Omega_1^+}} + Z_0 u_0^- \mathscr{L}_{|_{\Omega_1^-}}$. Applying the reduction map (4.20) we find that $\eta_0 = R_\# \mu_0$. In order to find the density also on the middle layer Ω_1^0 we look at two characterizations of $\mathcal{R}^{\zeta *}_{\mathrm{jump}}$. Computing the Legendre transform of $\mathcal{R}^{\zeta}_{\mathrm{jump}}$ given in (4.32) we immediately obtain

$$\mathcal{R}^{\zeta *}_{\mathrm{jump}}(\eta_0, [\![\xi]\!]) = \int_\Sigma \mathsf{a}(a,\zeta)\mathscr{C}^*([\![\xi]\!])\sqrt{u^- u^+}\,\mathrm{d}y.$$

On the other hand, by Lemma 4.18 we have that $\mathcal{R}^{\zeta}_{\mathrm{jump}}$ is given via a minimization problem, i.e., for $\dot{\eta}$ such that $\langle \varphi, \dot{\eta}\rangle = \int_\Sigma \tilde{\kappa}[\![\varphi]\!]\,\mathrm{d}y$ we have

$$\begin{aligned}\int_0^T \mathcal{R}^{\zeta}_{\mathrm{jump}}(\eta,\dot{\eta})\,\mathrm{d}t = \int_{[0,T]\times\overline{\Sigma}} \inf_{g_{y,t}} \frac{1}{2}\int_{I_1^0} a|\partial_d \log g_{y,t} - \partial_d \zeta|^2 g_{y,t} + \frac{\tilde{\kappa}^2}{a g_{y,t}}\,\mathrm{d}z\,\mathrm{d}N^0 \\ - \int_0^T \mathcal{R}^{\zeta *}_{\mathrm{jump}}(\eta, -[\![\log(g_{y,t})]\!])\,\mathrm{d}t\end{aligned}$$

subject to the boundary conditions $Z_0 u^\pm dydt = g_{y,t}(z^\pm)\,\mathrm{d}N^0$. Hence, the Legendre transform can be represented by the formula

$$
\begin{aligned}
\int_0^T \mathcal{R}^{\zeta*}_{\text{jump}}(\eta, [\![\xi]\!])\,\mathrm{d}t = \sup_{\tilde\kappa} \Bigg\{ \int_{[0,T]\times\overline{\Sigma}} \tilde\kappa [\![\xi]\!]\,\mathrm{d}N^0 \\
- \int_{[0,T]\times\overline{\Sigma}} \inf_{g_{y,t}} \frac{1}{2}\int_{I_1^0} a|\partial_d \log g_{y,t} - \partial_d\zeta|^2 g_{y,t} + \frac{\tilde\kappa^2}{a g_{y,t}}\,\mathrm{d}z\,\mathrm{d}N^0 \Bigg\} + C_0 \\
= C_0 - \frac{1}{2}\inf_{\tilde\kappa, g_{y,t}} \Bigg\{ \int_{[0,T]\times\overline{\Sigma}} \frac{1}{\mathrm{harm}_{I_1^0}(a g_{y,t})} \Big(\tilde\kappa - \mathrm{harm}_{I_1^0}(a g_{y,t})\,[\![\xi]\!]\Big)^2\,\mathrm{d}N^0 \\
+ \int_{[0,T]\times\overline{\Sigma}} \int_{I_1^0} a|\partial_d \log g_{y,t} - \partial_d\zeta|^2 g_{y,t} - \mathrm{harm}_{I_1^0}(a g_{y,t})\,[\![\xi]\!]^2\,\mathrm{d}z\,\mathrm{d}N^0 \Bigg\} \\
= C_0 - \frac{1}{2}\inf_{g} \Bigg\{ \int_{[0,T]\times\overline{\Sigma}} \int_{I_1^0} a|\partial_d \log g_{y,t} - \partial_d\zeta|^2 g_{y,t} \\
- \mathrm{harm}_{I_1^0}(a g_{y,t})\,[\![\xi]\!]^2\,\mathrm{d}z\,\mathrm{d}N^0 \Bigg\},
\end{aligned}
\tag{4.35}
$$

where we set $C_0 := \int_0^T \mathcal{R}^{\zeta*}_{\text{jump}}(\eta, -[\![\log(g_{y,t})]\!])\,\mathrm{d}t$. In the regular case

$$
\mathrm{d}N^0 = \left(\int_{I_1^0} Z_0 u_0(y,z,t)\,\mathrm{d}z\right)\mathrm{d}y\,\mathrm{d}t \quad \text{and} \quad g_{y,t}(z) = \frac{u_0(y,z,t)}{\int_{I_1^0} u_0(y,z,t)\,\mathrm{d}z},
$$

we have to minimize the function

$$
u \mapsto \int_{I_1^0} |\partial_d(\mathrm{E}_1'(u) - \zeta)|^2 a u\,\mathrm{d}z - \mathrm{harm}_{I_1^0}(au)\,[\![\xi_0]\!]^2
$$

subject to the boundary condition $u(y, z^\pm) = u_0^\pm(y)$. The minimizer is denoted by $U(u_0^\pm, [\![\xi_0]\!])$, is explicitly calculated in Lemma B.3 and is the desired density on the middle layer, i.e., we define the density on the whole domain Ω_1 via

$$
u(x) = \begin{cases} u_0^\pm(y) & \text{for } x \in \Omega_1^\pm, \\ U(u_0^\pm(y), [\![\xi_0(y)]\!])(z) & \text{for } x \in \Omega_1^0. \end{cases}
$$

Since the proof of Theorem 4.21 involves several approximations of u we outline the main ideas first.

Combined with the normalization $u_\varepsilon = \frac{u}{\int_{\Omega_1} u\,\mathrm{d}\pi_\varepsilon}$ we obtain that $\mu_\varepsilon = u_\varepsilon \pi_\varepsilon$ satisfies the mass constraint $\mu_\varepsilon(\Omega_1) = 1$. Note that $\int_{\Omega_1} u\,\mathrm{d}\pi_\varepsilon \to 1$ and hence, $\mu_\varepsilon \to \mu$. The continuity equation reads

$$
\langle \varphi, \dot\mu_\varepsilon \rangle = \int_{\Omega_1} \nabla'\varphi \cdot B\nabla'\xi_\varepsilon\,\mathrm{d}\mu_\varepsilon + \int_{\Omega_1^0} a\partial_d\varphi\partial_d\xi_\varepsilon\; u_\varepsilon\,\mathrm{d}x + \varepsilon^{-2}\int_{\Omega_1\setminus\Omega_1^0} a\partial_d\varphi\partial_d\xi_\varepsilon\; u_\varepsilon\,\mathrm{d}x.
$$

Passing to the limit in this continuity equation we obtain for test functions $\varphi \in \mathrm{C}^1(\overline{\Omega}_1)$ satisfying $(\partial_d\varphi)_{|_{\Omega_1^\pm}} \equiv 0$ that $\hat{\xi} = \lim \xi_\varepsilon$ satisfies $(\partial_d\hat{\xi})_{\Omega_1^\pm} \equiv 0$ and

$$\langle \varphi, \dot{\mu}_0\rangle = \int_\Sigma \Big\{ \sum_{\iota\in\{+,-\}} \nabla'\varphi^\iota \overline{B}_\iota \nabla'\hat{\xi}^\iota u^\iota \Big\} \mathrm{d}y + \int_{\Omega_1^0} a\partial_d\varphi\partial_d\hat{\xi}\; u\,\mathrm{d}x.$$

Testing with φ such that $\mathrm{supp}(\varphi) \subset \Omega_1^0$ we find that $a\partial_d\hat{\xi}\; u = \mathrm{harm}_{I_1^0}(au)\,[\![\hat{\xi}]\!]$ and consequently

$$\langle \varphi, \dot{\mu}_0\rangle = \int_\Sigma \Big\{ \sum_{\iota\in\{+,-\}} \nabla'\varphi^\iota \overline{B}_\iota \nabla'\hat{\xi}^\iota u^\iota \Big\} + \mathrm{harm}_{I_1^0}(au)\,[\![\hat{\xi}]\!][\![\varphi]\!]\,\mathrm{d}y.$$

Note that by Lemma B.3 we have that

$$\mathrm{harm}_{I_1^0}(au)\,[\![\xi_0]\!] = \mathsf{a}(a,\zeta)\mathscr{C}^{*\prime}([\![\xi_0]\!])\sqrt{u_0^+u_0^-}$$

with $\mathsf{a}(a,\zeta)$ given in (4.31). In particular, by uniqueness we find that $\hat{\xi} = \xi_0$ is the solution to (4.33). We recall

$$\mathcal{R}^*_{\mathrm{diff}}(\eta_0,\xi_0) = \int_\Sigma \frac{1}{2}\Big\{ \sum_{\iota\in\{+,-\}} \nabla'\xi_0^\iota \overline{B}_\iota \nabla'\xi_0^\iota u^\iota \Big\}\mathrm{d}y.$$

Hence

$$\begin{aligned}\limsup_{\varepsilon\downarrow 0} \mathcal{R}_\varepsilon(\mu_\varepsilon,\dot{\mu}_\varepsilon) &= \limsup_{\varepsilon\downarrow 0}\, \langle \xi_\varepsilon,\dot{\mu}_\varepsilon\rangle - \mathcal{R}^*_\varepsilon(\mu_\varepsilon,\xi_\varepsilon) = \langle \xi,\dot{\mu}\rangle - \liminf_{\varepsilon\downarrow 0} \mathcal{R}^*_\varepsilon(\mu_\varepsilon,\xi_\varepsilon) \\ &\le \langle \xi_0,\dot{\mu}\rangle - \mathcal{R}^*_{\mathrm{diff}}(\eta_0,\xi_0) - \frac{1}{2}\int_\Sigma \mathrm{harm}_{I_1^0}(au)\,[\![\xi_0]\!]^2\,\mathrm{d}y.\end{aligned}$$

Together with

$$\lim_{\varepsilon\downarrow 0} \mathcal{R}_\varepsilon\big(\mu_\varepsilon, -\mathrm{D}\mathcal{E}_\varepsilon(\mu_\varepsilon)\big) = \mathcal{R}^*_{\mathrm{diff}}\big(\eta_0, -\mathrm{D}\mathcal{E}_0(\eta_0)\big) + \int_{\Omega_1^0} \frac{1}{2}|\partial_d \mathrm{E}_1'(u)|^2 au\,\mathrm{d}x$$

we obtain that

$$\begin{aligned}&\limsup_{\varepsilon\downarrow 0} \Big(\mathcal{R}_\varepsilon(\mu_\varepsilon,\dot{\mu}_\varepsilon) + \mathcal{R}_\varepsilon\big(\mu_\varepsilon, -\mathrm{D}\mathcal{E}_\varepsilon(\mu_\varepsilon)+\zeta\big)\Big) \\ &\qquad\qquad \le \langle \xi_0,\dot{\mu}\rangle - \mathcal{R}^{\zeta*}_{\mathrm{eff}}(\mu,\xi_0) + \mathcal{R}^{\zeta*}_{\mathrm{eff}}\big(\mu, -\mathrm{D}\mathcal{E}_0(\mu)+\zeta\big).\end{aligned}$$

Here we used (4.35), i.e.,

$$\begin{aligned}&\int_{\Omega_1^0} \frac{1}{2}|\partial_d \mathrm{E}_1'(u) - \partial_d\zeta|^2 au\,\mathrm{d}x - \int_\Sigma \frac{1}{2}\,\mathrm{harm}_{I_1^0}(au)\,[\![\xi_0]\!]^2\,\mathrm{d}y \\ &\qquad\qquad = \mathcal{R}^{\zeta*}_{\mathrm{jump}}\big(\mu, [\![-\mathrm{D}\mathcal{E}_0(\mu)+\zeta]\!]\big) - \mathcal{R}^{\zeta*}_{\mathrm{jump}}\big(\mu, [\![\xi_0]\!]\big).\end{aligned}$$

Note that $\xi_0 \in \partial\mathcal{R}_{\text{eff}}(\mu,\dot\mu)$ and thus

$$\limsup_{\varepsilon\downarrow 0}\Big(\mathcal{R}_\varepsilon(\mu_\varepsilon,\dot\mu_\varepsilon)+\mathcal{R}_\varepsilon\big(\mu_\varepsilon,-\mathrm{D}\mathcal{E}_\varepsilon(\mu_\varepsilon)+\zeta\big)\Big)\le \mathcal{R}_{\text{eff}}(\mu,\dot\mu)+\mathcal{R}^*_{\text{eff}}\big(\mu,-\mathrm{D}\mathcal{E}_0(\mu)+\zeta\big).$$

In order to justify the steps above we need to modify the construction in two ways. Note that the solution $U(u_0^\pm,[\![\xi_0]\!])$ to the minimization problem given in (B.2) is not necessarily weakly differentiable in time hence, $\dot\mu_\varepsilon$ is not well defined. Moreover, the bound on the effective dissipation gives $U(u_0^\pm,[\![\xi_0]\!])\in \mathrm{L}^1\big((0,T)\times\Omega_1^0\big)$ only. To gain integrability we need to truncate $[\![\xi_0]\!]$.

In order to bound $\mathcal{R}_\varepsilon(\mu_\varepsilon,\dot\mu_\varepsilon)$ we discretize. Let $t_j=\frac{j}{n}T$ for $0\le j\le n$ and $t_j=0$ for $j<0$ and $t_j=T$ for $j>n$. We define the mean τ_j for $j\in\{0,\dots,n\}$ and the truncation χ_m for $m>0$ via

$$\tau_j h := \fint_{t_{j-1/2}}^{t_{j+1/2}} h\,\mathrm{d}t \quad\text{and}\quad \chi_m(\ell)=\max\big\{\min\{m,\ell\},-m\big\}.$$

We define $u^{(n,m)}$ at the nodal points $\{t_j\}$ via

$$u^{(n,m)}(t_j)=\begin{cases}\tau_j u_0^\pm & \text{on } \Omega_1^\pm,\\ U\big(\tau_j u_0^\pm,\chi_m([\![\tau_j\xi_0]\!])\big) & \text{on } \Omega_1^0.\end{cases}$$

and via affine interpolation in between. The recovery sequence is the given by $\mu_\varepsilon^{(n,m)}:=u_\varepsilon^{(n,m)}\pi_\varepsilon$ where $u_\varepsilon^{(n,m)}$ is the normalized density $u_\varepsilon^{(n,m)}=\frac{u^{(n,m)}}{\int_{\Omega_1}u^{(n,m)}\,\mathrm{d}\pi_\varepsilon}$.

Theorem 4.21. *Let $\alpha>0$ and $\eta_0\in\mathrm{L}^1\big((0,T);\mathcal{P}(\overline{\Omega}_1^0)\big)$ in the domain of $\mathfrak{D}_{\text{eff}}(\cdot,[0,T])$ with density u_0 with respect to the measure ϑ be such that $\alpha\le u_0\le\alpha^{-1}$ and $\dot\eta_0\in\mathrm{L}^2\big(0,T;\mathrm{H}^1(\Sigma)^*\times\mathrm{H}^1(\Sigma)^*\big)$. Then there exist sequences $n_\varepsilon,m_\varepsilon\to\infty$ as $\varepsilon\downarrow 0$ such that*

$$\limsup_{\varepsilon\downarrow 0}\mathfrak{D}_\varepsilon(\mu_\varepsilon^{(n_\varepsilon,m_\varepsilon)},[0,T])\le\mathfrak{D}_{\text{eff}}(\mu_0,[0,T]).$$

Proof. For the limit passage in the slope term (step 3,5 and 7) we use that $\mathcal{R}^*_\varepsilon\big(\mu_\varepsilon^{(n,m)},-\mathrm{D}\mathcal{E}_\varepsilon(\mu_\varepsilon^{(n,m)})+\zeta\big)-\mathcal{R}^*_\varepsilon\big(\mu_\varepsilon^{(n,m)},-\mathrm{D}\mathcal{E}_\varepsilon(\mu_\varepsilon^{(n,m)})\big)$ depends linearly on $u_\varepsilon^{(n,m)}$. Hence, it suffices to pass to the limit in $\mathcal{R}^*_\varepsilon\big(\mu_\varepsilon^{(n,m)},-\mathrm{D}\mathcal{E}_\varepsilon(\mu_\varepsilon^{(n,m)})\big)$ only.

Step 1: We show that $\int_0^T\mathcal{R}_\varepsilon(\mu_\varepsilon^{(n,m)},\dot\mu_\varepsilon^{(n,m)})\,\mathrm{d}t$ is bounded. Note that by Lemma B.3 we have that $\alpha\le u^{(n,m)}\le 2\alpha^{-1}\big(1+\cosh(m/2)\big)\mathrm{e}^{\|\zeta\|_\infty}$. In particular,

$$\int_{\Omega_1}(\dot u^{(n,m)})^2\mathbf{m}_\varepsilon\,\mathrm{d}\Pi_\varepsilon\le\frac{4\alpha^{-2}\big(1+\cosh(m/2)\big)^2\mathrm{e}^{2\|\zeta\|_\infty}n^2}{T^2}.$$

The normalization constant $\big(\int_{\Omega_1}u^{(n,m)}\,\mathrm{d}\pi_\varepsilon\big)^{-1}$ is bounded in $\mathrm{W}^{1,\infty}(0,T)$ and converges strongly to 1 in $\mathrm{W}^{1,p}(0,T)$ for any $1\le p<\infty$. Hence, $\mathcal{R}_\varepsilon(\mu_\varepsilon^{(n,m)},\dot\mu_\varepsilon^{(n,m)})$ is bounded in $\mathrm{L}^\infty(0,T)$ as $\varepsilon\downarrow 0$ (see Lemma 4.20).

Step 2. Passing to the ε-limit in the continuity equation: Note that as $\varepsilon\downarrow 0$ the limit measure $\mu^{(n)}$ does only depend on n. Only the limit of the densities on the middle layer depend on m. Thus the limit of the solutions to the continuity equations $\xi_\varepsilon^{(n,m)} \in \partial\mathcal{R}_\varepsilon(\mu_\varepsilon^{(n,m)}, \dot\mu_\varepsilon^{(n,m)})$ also depend on n and m. The limit exists since $\mathcal{R}_\varepsilon(\mu_\varepsilon^{(n,m)}, \dot\mu_\varepsilon^{(n,m)})$ is bounded, or equivalently $\|\xi_\varepsilon^{(n,m)}\|_{\mathrm{W}_\varepsilon(u_\varepsilon^{(n,m)})}$ is bounded in $\mathrm{L}^\infty(0,T)$. Hence, we have weak convergence of $\xi^{(n,m)}_{\varepsilon|_{\Omega_1\setminus\Omega_1^0}}$ to$\xi^{(n,m)}$ in $\mathrm{L}^\infty\big(0,T;\mathrm{H}^1(\Omega_1\setminus\Omega_1^0)\big)$ and bounded mass of $\xi^{(n,m)}_{\varepsilon|_{\Omega_1^0}}$ (cf Lemma 4.4). Since $\dot u_\varepsilon^{(n,m)} \to \dot u^n$ strongly in $\mathrm{L}^2\big((0,T)\times\Omega_1\big)$ we obtain

$$\int_0^T \langle \xi_\varepsilon^{(n,m)}, \dot\mu_\varepsilon^{(n,m)}\rangle \,\mathrm{d}t \to \int_0^T \langle \xi^{(n,m)}, \dot\mu^n\rangle \,\mathrm{d}t$$

Moreover, since $\|\xi_\varepsilon^{(n,m)}\|_{\mathrm{W}_\varepsilon(u_\varepsilon^{(n,m)})}$ is bounded in $\mathrm{L}^\infty(0,T)$ we have additionally that $(\partial_d\xi^{(n,m)})_{|_{\Omega_1^\pm}} \equiv 0$. Exploiting that $\dot\mu^{(n,m)} = 0$ on Ω_1^0 and passing to the limit in the continuity equation $\xi_\varepsilon^{(n,m)} \in \partial\mathcal{R}_\varepsilon(\mu_\varepsilon^{(n,m)}, \dot\mu_\varepsilon^{(n,m)})$ (see equation 4.15) we conclude that

$$\langle \varphi, \dot\mu_0^{(n)}\rangle = \int_\Sigma \Big\{ \sum_{\iota\in\{+,-\}} \nabla'\varphi^\iota \overline{B}_\iota \nabla'\xi^{\iota,n,m}\; u^{\iota,n} \Big\} + \mathrm{harm}_{I_1^0}\big(au^{(n,m)}\big)\, [\![\varphi]\!][\![\xi^{(n,m)}]\!]\,\mathrm{d}y.$$

Hence, we obtain the estimate

$$\begin{aligned}\liminf_{\varepsilon\downarrow 0} \int_0^T \mathcal{R}_\varepsilon^*(\mu_\varepsilon^{(n,m)}, \xi_\varepsilon^{(n,m)})\,\mathrm{d}t \geq &\int_0^T \mathcal{R}^*_{\mathrm{diff}}(\eta_0^{(n)}, \xi^{(n,m)})\,\mathrm{d}t \\ &+ \int_0^T \int_\Sigma \frac{1}{2}\,\mathrm{harm}_{I_1^0}\big(au^{(n,m)}\big)\, [\![\xi^{(n,m)}]\!]^2\,\mathrm{d}y\,\mathrm{d}t\end{aligned}$$

Here we used the identification $\mu_0^{(n)} \widehat{=} \eta_0^{(n)}$ since the density of $\mu_0^{(n)}$ does not depend on the vertical variable z.

Step 3. Passing to the ε-limit in the slope: We recall the mean

$$\tau_j h = ⨍_{t_{j-1/2}}^{t_{j+1/2}} h\,\mathrm{d}t.$$

By Jensen's estimate with respect to measure $u\,\mathrm{d}t$ we observe

$$\tau_j\left(\int_\Sigma |\nabla u^\pm|^2 \frac{1}{u^\pm}\,\mathrm{d}y\right) \geq \int_\Sigma |\nabla\tau_j u^\pm|^2 \frac{1}{\tau_j u^\pm}\,\mathrm{d}y.$$

Since for $t\in(t_j,t_{j+1})$ we have that $u^{(n)}(t)$ is a convex combination of $u^{(n)}(t_j)$ and $u^{(n)}(t_{j+1})$ we obtain by convexity of the slope term that

$$\int_0^T \int_\Sigma |\nabla u^{\pm,(n)}(t)|^2 \frac{1}{u^{\pm,(n)}(t)}\,\mathrm{d}y\,\mathrm{d}t \leq \int_0^T \int_\Sigma |\nabla u^{\pm}(t)|^2 \frac{1}{u^{\pm}(t)}\,\mathrm{d}y\,\mathrm{d}t.$$

Hence, the integrals on the bottom and top layers are bounded. On the middle layer we assume for simplicity $a \in \mathrm{C}^1(\Omega_1^0)$. Hence, $\nabla'\big(U\big(\tau_j u_0^\pm, \chi_m(\llbracket \tau_j \xi_0 \rrbracket)\big)\big) \in \mathrm{L}^2\big((0,T)\times\Omega_1^0\big)$ since $\nabla'\llbracket \xi_0 \rrbracket, \nabla' u^\pm \in \mathrm{L}^2\big((0,T)\times\Omega_1^0\big)$ and $|\sinh(\chi_m(\llbracket \tau_j \xi_0 \rrbracket/2)| \leq \sinh(m/2)$ and $\alpha \leq u^\pm \leq \alpha^{-1}$. Thus we obtain

$$\lim_{\varepsilon \downarrow 0} \int_0^T \mathcal{R}_\varepsilon^*\big(\mu_\varepsilon^{(n)}, -\mathrm{D}\mathcal{E}_\varepsilon(\mu_\varepsilon^{(n)})\big)\,\mathrm{d}t = \int_0^T \mathcal{R}_{\mathrm{diff}}^*\big(\eta_0^{(n)}, -\mathrm{D}\mathcal{E}_0(\eta_0^{(n)})\big) + \int_{\Omega_1^0} \frac{1}{2}|\partial_d \mathrm{E}_1'(u^{(n,m)})|^2 a u^{(n,m)}\,\mathrm{d}x \Big\}\,\mathrm{d}t$$

Of course, the assumption $a \in \mathrm{C}^1(\Omega_1^0)$ can be canceled by an approximation a_ε such that $\varepsilon^{\delta/4}\nabla' a_\varepsilon \to 0$ and thus $\varepsilon^{\delta/2}\nabla'\big(U\big(\tau_j u_0^\pm, \chi_m(\llbracket \tau_j \xi_0 \rrbracket)\big)\big) \to 0$.

Step 4. Passing to the n-limit in the continuity equation: We have strong convergence $\dot{u}^{(n)} \to \dot{u}$ in $\mathrm{L}^2\big(0,T;\mathrm{H}^1(\Sigma)^* \times \mathrm{H}^1(\Sigma)^*\big)$ by Lemma D.5.

Consequently, we obtain that $\xi^{(n,m)}$ is bounded in $\mathrm{L}^2\big(0,T;\mathrm{H}^1(\Sigma)\times\mathrm{H}^1(\Sigma)\big)$, since for φ with $\int_\Sigma \varphi^+ + \varphi^- \,\mathrm{d}y = 0$ we have

$$|\!|\!|\varphi|\!|\!|^2 := \mathcal{R}_{\mathrm{diff}}^*(\mu^{(n)}, \varphi) + \int_\Sigma \mathrm{harm}_{I_1^0}\left(a u^{(n,m)}\right) \llbracket \varphi \rrbracket^2 \,\mathrm{d}y$$

is equivalent to $\|\varphi\|^2_{\mathrm{H}^1(\Sigma)\times\mathrm{H}^1(\Sigma)}$. Hence, there exists a weak limit $\xi^{(m)}$ of $\xi^{(n,m)}$ in $\mathrm{L}^2\big(0,T;\mathrm{H}^1(\Sigma)\times\mathrm{H}^1(\Sigma)\big)$. By dominated convergence theorem we obtain strong convergence $u^{(n,m)}_{|(0,T)\times\Omega_1^0} \to U\big(u_0^\pm, \chi_m(\llbracket \xi_0 \rrbracket)\big) =: u^{(m)}$ in $\mathrm{L}^p(0,T\times\Omega_1^0)$ for any $p \geq 1$. In particular, $\mathrm{harm}_{I^0}\left(a u^{(n,m)}\right) \to \mathrm{harm}_{I^0}\left(a u^{(m)}\right)$ in $\mathrm{L}^p(0,T\times\Sigma)$ for any $p \geq 1$. Hence, we obtain

$$\langle \varphi, \dot{\eta}_0 \rangle = \int_\Sigma \Big\{ \sum_{\iota \in \{+,-\}} \nabla' \varphi^\iota \overline{B}_\iota \nabla' \xi^{\iota,m}\ u_0^\iota \Big\} + \mathrm{harm}_{I^0}\left(a u^m\right) \llbracket \varphi \rrbracket \llbracket \xi^m \rrbracket \,\mathrm{d}y$$

and

$$\liminf_{n\to\infty} \int_0^T \Big\{ \mathcal{R}_{\mathrm{diff}}^*(\eta_0^{(n)}, \xi^{(n,m)}) + \int_\Sigma \frac{1}{2}\mathrm{harm}_{I^0}\left(a u^{(n,m)}\right) \llbracket \xi^{(n,m)} \rrbracket^2 \,\mathrm{d}y \Big\}\,\mathrm{d}t \geq \int_0^T \Big\{ \mathcal{R}_{\mathrm{diff}}^*(\eta_0, \xi^{(m)}) + \int_\Sigma \frac{1}{2}\mathrm{harm}_{I^0}\left(a u^{(m)}\right) \llbracket \xi^{(m)} \rrbracket^2 \,\mathrm{d}y \Big\}\,\mathrm{d}t$$

Step 5. Passing to the n-limit in the slope: By strong convergence we immediately obtain

$$\lim_{n\to\infty} \int_0^T \Big\{ \mathcal{R}_{\mathrm{diff}}^*\big(\eta_0^{(n)}, -\mathrm{D}\mathcal{E}_0(\eta_0^{(n)})\big) + \int_{\Omega_1^0} \frac{1}{2}|\partial_d \mathrm{E}_1'(u^{(n,m)})|^2 a u^{(n,m)}\,\mathrm{d}x \Big\}\,\mathrm{d}t = \int_0^T \Big\{ \mathcal{R}_{\mathrm{diff}}^*\big(\eta_0, -\mathrm{D}\mathcal{E}_0(\eta_0)\big) + \int_{\Omega_1^0} \frac{1}{2}|\partial_d \mathrm{E}_1'(u^m)|^2 a u^m\,\mathrm{d}x \Big\}\,\mathrm{d}t$$

Step 6. Passing to the m-limit in the continuity equation: Note that $u^m \nearrow U\big(u_0^\pm, [\![\xi_0]\!]\big) =: u^0$ monotonously. Moreover, since $\dot{u}_0 \in \mathrm{L}^2\big(0,T;\mathrm{H}^1(\Sigma)^* \times \mathrm{H}^1(\Sigma)^*\big)$ and $\alpha \le u$ we have

$$\|\dot{u}_0\|^2_{\mathrm{L}^2\left(0,T;\left(\mathrm{H}^1(\Sigma)\times\mathrm{H}^1(\Sigma)\right)^*\right)} \ge c\int_0^T\int_\Sigma |\nabla\xi^{+,m}|^2 + |\nabla\xi^{-,m}|^2 + \mathrm{harm}_{I^0}\left(au^{(m)}\right)[\![\xi^{(m)}]\!]^2\,\mathrm{d}y\,\mathrm{d}t.$$

Hence, there exists a weak limit ξ in $\mathrm{L}^2\big(0,T;\mathrm{H}^1(\Sigma)\times\mathrm{H}^1(\Sigma)\big)$. In particular,

$$\lim_{m\to\infty}\int_0^T \langle \xi^m, \dot{\eta}_0\rangle\,\mathrm{d}t = \int_0^T \langle \xi, \dot{\eta}_0\rangle\,\mathrm{d}t$$

and

$$\begin{aligned}\liminf_{m\to\infty}\int_0^T\int_\Sigma &\mathrm{harm}_{I^0}\left(au^{(m)}\right)[\![\xi^{(m)}]\!]^2\,\mathrm{d}y\,\mathrm{d}t\\ &\ge \lim_{m_2\to\infty}\liminf_{m_1\to\infty}\int_0^T\int_\Sigma \mathrm{harm}_{I^0}\left(au^{(m_2)}\right)[\![\xi^{(m_1)}]\!]^2\,\mathrm{d}y\,\mathrm{d}t\\ &\ge \int_0^T\int_\Sigma \mathrm{harm}_{I^0}\left(au^0\right)[\![\xi]\!]^2\,\mathrm{d}y\,\mathrm{d}t\end{aligned}$$

as well as

$$\liminf_{m\to\infty}\int_0^T \mathcal{R}^*_{\mathrm{diff}}(\eta_0,\xi^{(m)})\,\mathrm{d}t \ge \int_0^T \mathcal{R}^*_{\mathrm{diff}}(\eta_0,\xi)\,\mathrm{d}t.$$

Since $\mathrm{harm}_{I^0}\left(au^{(m)}\right)[\![\xi^{(m)}]\!]^2$ is bounded, we have weak convergence in the sense of [AGS05, Thm 5.4.4] (see Section 2.2) $\mathrm{harm}_{I^0}\left(au^{(m)}\right)[\![\xi^{(m)}]\!] \rightharpoonup \mathrm{harm}_{I^0}(au)[\![\xi']\!]$ and in $\mathrm{L}^1\big((0,T)\times\Sigma\big)$. By testing with the strong convergent sequence

$$\varphi_m = \varphi\frac{\mathrm{harm}_{I^0}(au^m)}{\mathrm{harm}_{I^0}(au^0)}$$

we obtain that $[\![\xi']\!]$ coincides with the weak L^2-limit $[\![\xi]\!]$. Passing to the limit in the continuity equation we obtain

$$\int_0^T\langle\varphi,\dot{\eta}_0\rangle\,\mathrm{d}t = \int_0^T\int_\Sigma\left\{\sum_{\iota\in\{+,-\}}\nabla'\varphi^\iota\overline{B}_\iota\nabla'\xi^\iota\ u^\iota\right\} + \mathrm{harm}_{I^0}\left(au^0\right)[\![\varphi]\!][\![\xi]\!]\,\mathrm{d}y\,\mathrm{d}t.$$

By Lemma B.3 we know that

$$\mathrm{harm}_{I^0}\left(au^0\right)[\![\xi_0]\!] = \mathsf{a}(a,\zeta)\mathscr{C}^{*'}([\![\xi_0]\!])\sqrt{u^+u^-}.$$

By uniqueness we conclude $\xi = \xi_0$.

Step 7. Passing to the m-limit in the slope: It remains to show that

$$\lim_{m\to\infty}\int_0^T\int_{\Omega_1^0}\frac{1}{2}|\partial_d\mathrm{E}_1'(u^m)|^2au^m\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_{\Omega_1^0}\frac{1}{2}|\partial_d\mathrm{E}_1'(u^0)|^2au^0\,\mathrm{d}x\,\mathrm{d}t.$$

Note that Lemma B.3 gives

$$\int_{I^0} \frac{1}{2}|\partial_d \mathrm{E}_1'(u^m) + \partial_d\zeta|^2 a u^m \,\mathrm{d}z$$
$$= \mathsf{a}(a.\zeta)\Big(\mathscr{C}^*([\![-\log u + \zeta]\!]) - \mathscr{C}^*(\chi_m[\![\xi_0]\!]) + \mathscr{C}^{*\prime}(\chi_m[\![\xi_0]\!])\chi_m[\![\xi_0]\!]\Big)\sqrt{u^+u^-}.$$

We conclude Step 7 since both, $\mathscr{C}^*(\chi_m[\![\xi_0]\!])$ and $\mathscr{C}^{*\prime}(\chi_m[\![\xi_0]\!])\chi_m[\![\xi_0]\!]$ are integrable and converge monotonously.

Step 8: Conclusion. From steps 1-7 we conclude with $\xi_0 \in \partial\mathcal{R}^\zeta(\mu_0.\dot\mu_0)$ that

$$\limsup_{m\to\infty}\limsup_{n\to\infty}\limsup_{\varepsilon\downarrow 0}\mathfrak{D}_\varepsilon^\zeta(\mu_\varepsilon^{(n,m)},[0,T])$$
$$\leq \int_0^T \Big\{ \langle \xi_0, \dot\mu_0\rangle - \mathcal{R}^*_{\mathrm{div}}(\mu_0,\xi_0) - \mathcal{R}^{\zeta*}_{\mathrm{memb}}(\mu_0,[\![\xi_0]\!]) + \mathcal{R}^*_{\mathrm{div}}(\mu_0, -\mathrm{D}\mathcal{E}_0(\mu_0)+\zeta)$$
$$+ \mathcal{R}^{\zeta*}_{\mathrm{memb}}(\mu_0,[\![-\mathrm{D}\mathcal{E}_0(\mu_0)+\zeta]\!])\Big\}\,\mathrm{d}t = \mathfrak{D}^\zeta_{\mathrm{eff}}(\mu_0,[0,T]).$$

With Lemma 1.2 we conclude. □

4.2.4 Convergence of the gradient flows

In the sequel, we consider solutions to the EDB only and show that their limit is the solution to the gradient flow induced by $(\mathscr{X},\mathcal{E}_0,\mathcal{R}_{\mathrm{eff}})$.

Note that in Subsection 4.2.1 we obtained two limits of u_ε. There is the pointwise limit of $R_\#\mu_\varepsilon$ denoted by $\hat\eta = \hat u\vartheta$. And there is the $\mathrm{L}^1\big(0,T;\mathrm{W}^{1,1}(\Omega_1^\pm)\big)$ limit of $u_{\varepsilon|_{\Omega_1^\pm}}$ denoted by $u^\pm$ with $\partial_d u^\pm \equiv 0$ which corresponds to $\eta = u\vartheta$. Of course, both limits coincide almost everywhere in $[0,T]\times\Sigma$. However, this fact needs to be taken into account when passing to the limit in (EDB) since we have the pointwise evaluation $\mathcal{E}_\varepsilon\big(\mu_\varepsilon(t)\big)$. Fortunately, we have a.e. convergence of the energies (cf. Lemma 4.13) and $\mathfrak{D}_{\mathrm{eff}}$ is of integral form, i.e., the Γ-limit of $\mathfrak{D}_\varepsilon(\cdot,[t_1,t_2])$ is $\mathfrak{D}_{\mathrm{eff}}(\cdot,[t_1,t_2])$. Choosing t_j such that $\mathcal{E}_\varepsilon\big(\mu_\varepsilon(t_j)\big) \to \mathcal{E}_0\big(\eta(t_j)\big)$ we conclude

$$0 = \liminf \mathcal{E}_\varepsilon\big(\mu_\varepsilon(t_2)\big) - \mathcal{E}_\varepsilon\big(\mu_\varepsilon(t_1)\big) + \mathfrak{D}_\varepsilon(\mu_\varepsilon,[t_1,t_2])$$
$$\geq \mathcal{E}_0\big(\eta(t_2)\big) - \mathcal{E}_0\big(\eta(t_1)\big) + \mathfrak{D}_{\mathrm{eff}}(\eta,[t_1,t_2]) \quad (4.36)$$

Note that it is not obvious (using only the information provided by $\mathcal{E}_\varepsilon$ and $\mathcal{R}_\varepsilon$) whether

$$\lim_{\varepsilon\downarrow 0}\mu_\varepsilon(0) = \hat\eta(0) \stackrel{!}{=} \eta(0) = \lim_{t\downarrow 0}\eta(t).$$

Using the chain-rule we conclude that η is indeed the gradient flow induced by $(\mathscr{X},\mathcal{E}_0,\mathcal{R}_{\mathrm{eff}})$. For $\Sigma\subset\mathbb{R}^1$ the chain-rule is proven in the following lemma.

Lemma 4.22. *Let $\Sigma \subset \mathbb{R}^1$ be an interval and let*

$$\mathfrak{D}_{\mathrm{eff}}(\eta,[0,T]) < \infty \quad \textit{and} \quad \sup_{t\in[0,T]} |\mathcal{E}_0(\eta(t))| < \infty.$$

Then the chain-rule holds, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_0\big(\eta(t)\big) = \langle \mathrm{D}\mathcal{E}_0\big(\eta(t)\big), \dot\eta(t)\rangle \quad \textit{for a.a. } t \in (0,T).$$

Proof. We aim to apply [MRS13, Prop. 2.4]. Therefore we need to show that $\eta \in \mathrm{AC}([0,T];\mathscr{B})$ and $\mathrm{D}\mathcal{E}_0(\eta) \in \mathrm{L}^1(0,T;\mathscr{B}^*)$ for the reflexive Banach space $\mathscr{B} = Y^*$ where $Y = \{\xi \in \mathrm{H}^1(\Sigma)\times\mathrm{H}^1(\Sigma) \,|\, \int_\Sigma \xi^+ + \xi^- \,\mathrm{d}y = 0\}$ equipped with the norm $\|\xi\|_Y = \|\nabla\xi^-\|_{\mathrm{L}^2(\Sigma)} + \|\nabla\xi^+\|_{\mathrm{L}^2(\Sigma)} + \|[\![\xi]\!]\|_{\mathrm{L}^{\mathscr{C}^*}(\Sigma)}$. Note that $\|\cdot\|_Y$ is equivalent to $\|\cdot\|_{\mathrm{H}^1(\Sigma)\times\mathrm{H}^1(\Sigma)}$ on Y since $\mathrm{H}^1(\Sigma) \subset \mathrm{C}^0(\Sigma)$.

Step 1: In the following, we show $\eta \in \mathrm{AC}([0,T];\mathscr{B})$. Note that we have that $\sqrt{u^\pm} \in \mathrm{L}^2\big(0,T;\mathrm{H}^1(\Sigma)\big)$. In particular, $u^\pm \in \mathrm{L}^1\big(0,T;\mathrm{C}^0(\Sigma)\big)$. We estimate

$$\begin{aligned}\int_{t_1}^{t_2}\int_\Sigma w^\pm \overline{B}_\pm \nabla\varphi \; u^\pm \,\mathrm{d}y\,\mathrm{d}t &\le c\|w^\pm\|_{\mathrm{L}^2(u^\pm\,\mathrm{d}y\,\mathrm{d}t)}\left(\int_{t_1}^{t_2}\int^\Sigma |\nabla\varphi|^2 u^\pm \,\mathrm{d}y\,\mathrm{d}t\right)^{1/2}\\ &\le c\|w^\pm\|_{\mathrm{L}^2(u^\pm\,\mathrm{d}y\,\mathrm{d}t)}\|\nabla\varphi\|_{\mathrm{L}^2(\Sigma)}\left(\int_{t_1}^{t_2}\|u^\pm\|_{\mathrm{L}^\infty}\,\mathrm{d}t\right)^{1/2}\end{aligned}$$

and

$$\begin{aligned}\int_{t_1}^{t_2}\int_\Sigma \kappa[\![\varphi]\!]\sqrt{u^+u^-}\,\mathrm{d}y\,\mathrm{d}t \le \|[\![\varphi]\!]\|_{\mathrm{L}^{\mathscr{C}^*}(\Sigma)}\bigg(&\int_{t_1}^{t_2}\int_\Sigma \mathscr{C}(\kappa)\sqrt{u^+u^-}\,\mathrm{d}y\,\mathrm{d}t\\ &+\int_{t_1}^{t_2}\sqrt{\|u^+\|_{\mathrm{L}^\infty}\;\|u^-\|_{\mathrm{L}^\infty}}\,\mathrm{d}t\bigg).\end{aligned}$$

Here we used for $k = \|[\![\varphi]\!]\|_{\mathrm{L}^{\mathscr{C}^*}(\Sigma)}$ that

$$\int_\Sigma \mathscr{C}^*([\![\varphi]\!]/k)\,\mathrm{d}y \le 1.$$

Thus, we have shown, that

$$|\langle \eta(t_2) - \eta(t_1), \varphi\rangle| \le \|\varphi\|_Y \int_{t_1}^{t_2} m(\tau)\,\mathrm{d}\tau$$

for $m \in \mathrm{L}^1(0,T)$ depending on $w^\pm, \kappa$ and $u^\pm$, i.e., $\eta \in \mathrm{AC}([0,T]:\mathscr{B})$.

Step 2: In the following, we show $\mathrm{D}\mathcal{E}_0(\eta) \in \mathrm{L}^1(0,T;\mathscr{B}^*)$ for $u \ge \alpha$. We abbreviate $\xi^\pm = u^\pm \log u^\pm$. By the bound on the (dual) dissipation it immediately follows that

$$\infty > \int_0^T\int_\Sigma |\nabla\xi^\pm|^2 u^\pm \,\mathrm{d}y\,\mathrm{d}t \ge \int_0^T\int_\Sigma |\nabla\xi^\pm|^2\alpha\,\mathrm{d}y\,\mathrm{d}t$$

and

$$\infty > \int_0^T \int_\Sigma \mathscr{C}^*([\![\xi]\!])\sqrt{u^+u^-}\,\mathrm{d}y\,\mathrm{d}t \geq \int_0^T \int_\Sigma \mathscr{C}^*([\![\xi]\!])\alpha\,\mathrm{d}y\,\mathrm{d}t.$$

Using $\max\left\{1, \int_\Sigma \mathscr{C}^*([\![\xi]\!])\,\mathrm{d}y\right\} \geq \|[\![\xi]\!]\|_{\mathrm{L}^{\mathscr{C}^*}(\Sigma)}$ we conclude Step 2.

For $u \geq \alpha$ we obtain the chain-rule by [MRS13, Prop. 2.4].

Step 3: In the general case $u \geq 0$, we verify that for $u_\alpha = u + \alpha$ we have

$$\mathcal{R}_{\text{eff}}(\eta_\alpha, \dot{\eta}_\alpha) \leq \mathcal{R}_{\text{eff}}(\eta, \dot{\eta}) \quad \text{and} \quad \mathcal{R}^*_{\text{eff}}\big(\eta_\alpha, \log(\eta_\alpha)\big) \leq \mathcal{R}^*_{\text{eff}}\big(\eta, \mathrm{D}\mathcal{E}_0(\eta)\big)$$

as follows. Note that $\mathcal{R}^*_{\text{eff}}(\eta_\alpha, \xi) \geq \mathcal{R}^*_{\text{eff}}(\eta, \xi)$ hence, $\mathcal{R}_{\text{eff}}(\eta_\alpha, \dot{\eta}_\alpha) \leq \mathcal{R}_{\text{eff}}(\eta, \dot{\eta})$ since $\dot{\eta} = \dot{\eta}_\alpha$. The second estimate follows from $\mathscr{C}^*([\![\log u]\!])\sqrt{u^+u^-} = (\sqrt{u^+} - \sqrt{u^-})^2$ and $|\nabla \log u|^2 u = |\nabla u|^2/u$.

Moreover, we have $v_\alpha^\pm = v^\pm u^\pm / u_\alpha^\pm$ and $\kappa_\alpha = \kappa\sqrt{u^-u^+}/\sqrt{u_\alpha^- u_\alpha^+}$. Hence

$$|\kappa_\alpha [\![\log u_\alpha]\!] \sqrt{u_\alpha^- u_\alpha^+}| \leq |\kappa [\![\log u]\!] \sqrt{u^-u^+}|$$

and

$$|v_\alpha^\pm \nabla \log u_\alpha^\pm | u_\alpha^\pm \leq |v^\pm \nabla \log u^\pm | u^\pm.$$

hence, the integrand of $\langle \mathrm{D}\mathcal{E}_0\big(\eta_\alpha(t)\big), \dot{\eta}_\alpha(t)\rangle$ is dominated by the integrand of its α-limit $\langle \mathrm{D}\mathcal{E}_0\big(\eta(t)\big), \dot{\eta}(t)\rangle$. Hence, we pass to the limit $\alpha \downarrow 0$ in the chain-rule and conclude. □

Hence, we conclude that the limit η is a solution to the gradient flow, i.e., for almost all $t \in (0, T)$ we have $-\mathrm{D}\mathcal{E}_0(\eta) \in \partial\mathcal{R}_{\text{eff}}(\eta, \dot{\eta})$. Moreover, if

$$\operatorname{harm}\left(a\mathrm{e}^\zeta\right)\sqrt{\mathrm{e}^{-(\zeta^+ + \zeta^-)}} = \operatorname{harm}(a)$$

then $\mathfrak{D}_\varepsilon^\zeta$ converges to $\mathfrak{D}_{\text{eff}}^\zeta$ with the effective dissipation potential independent of ζ.

The limit equation reads

$$\dot{u}^\pm = \operatorname{div}(\overline{B}^\pm \nabla u^\pm) - \operatorname{harm}_{I_1^0}(a)\,(u^\pm - u^\mp)$$

with homogenous Neumann boundary conditions $\overline{B}^\pm \nabla u^\pm \cdot \nu = 0$ on $\partial\Sigma$.

5 Transmission condition in a porous medium equation

This chapter deals with gradient systems inducing the porous medium equation

$$\dot{\mathbf{u}} = \partial_x(\tilde{\mathbf{a}}_\varepsilon \partial_x \mathbf{u}^{\tilde{m}})$$

on the domain $\Omega_\varepsilon =]-1+\varepsilon x_-, 1+\varepsilon x_+[$ with $x_+ - x_- = 1$ and homogenous Neumann boundary conditions. We consider the following two gradient systems. First,

$$\begin{aligned}
\mathbf{X}^{(1)}_\varepsilon &= \mathcal{P}(\overline{\Omega}_\varepsilon), \\
\hat{\mathcal{E}}^{(1)}_\varepsilon(\boldsymbol{\mu}) &= \begin{cases} \int_{\Omega_\varepsilon} \mathrm{E}_m(\mathbf{u})\,\mathrm{d}\boldsymbol{\pi}_\varepsilon & \text{if } \boldsymbol{\mu} = \mathbf{u}\boldsymbol{\pi}_\varepsilon, \\ \infty & \text{else,} \end{cases} \\
\hat{\mathcal{R}}^{(1)*}_\varepsilon(\boldsymbol{\mu}, \boldsymbol{\xi}) &= \frac{1}{2}\int_{\Omega_\varepsilon} \mathbf{a}_\varepsilon |\partial_x \boldsymbol{\xi}|^2 \mathsf{m}(\mathbf{u})\,\mathrm{d}\boldsymbol{\pi}_\varepsilon,
\end{aligned} \tag{5.1a}$$

where $\boldsymbol{\pi}_\varepsilon$ is the normalized Lebesgue measure on Ω_ε, $\boldsymbol{\mu} = \mathbf{u}\boldsymbol{\pi}_\varepsilon + \boldsymbol{\pi}^\perp_\varepsilon$, $\mathrm{E}_m(u) = \frac{u^m - u}{m-1} - u + 1$ and the mobility function $\mathsf{m} : [0,\infty) \to [0,\infty)$ is concave. With $0 < \gamma \leq 1$ and $\mathsf{m}(u) = u^\gamma$ we have $\tilde{m} = m + \gamma - 1$ and $\tilde{\mathbf{a}}_\varepsilon \tilde{m} = \mathbf{a}_\varepsilon m$. The second gradient system is given by

$$\begin{aligned}
\mathbf{X}^{(2)}_\varepsilon &= \mathcal{P}(\overline{\Omega}_\varepsilon), \\
\hat{\mathcal{E}}^{(2)}_\varepsilon(\boldsymbol{\mu}) &= \begin{cases} \int_{\Omega_\varepsilon} \gamma \mathrm{E}_1(\mathbf{u})\,\mathrm{d}\boldsymbol{\pi}_\varepsilon & \text{if } \boldsymbol{\mu} = \mathbf{u}\boldsymbol{\pi}_\varepsilon, \\ \infty & \text{else,} \end{cases} \\
\hat{\mathcal{R}}^{(2)*}_\varepsilon(\boldsymbol{\mu}, \boldsymbol{\xi}) &= \frac{1}{2}\int_{\Omega_\varepsilon} \mathbf{a}_\varepsilon |\partial_x \boldsymbol{\xi}|^2 \mathbf{u}^\gamma\,\mathrm{d}\boldsymbol{\pi}_\varepsilon.
\end{aligned} \tag{5.1b}$$

As pointed out in Section 2.1 the gradient system (5.1a) is very closely related to the metric gradient flow formulation with respect to the Wasserstein metric with a nonlinear mobility (see [DNS09, CLSS10]). The gradient system (5.1b) is related to [DSZ16] since $\gamma \mathrm{E}_1'(u) = \log(u^\gamma)$. However, (5.1b) cannot be related to the Wasserstein space with nonlinear (and concave) mobility since $\gamma > 1$. Note that linear tilts affect the resulting equation differently for each gradient system. We compute

$$\dot{\boldsymbol{\mu}} \in \partial \hat{\mathcal{R}}^{(1)*}_\varepsilon\big(\boldsymbol{\mu}, -\mathrm{D}\hat{\mathcal{E}}^{(1)}_\varepsilon(\boldsymbol{\mu}) - \zeta\big) \iff \dot{u} = \partial_x\left(a_\varepsilon \frac{m}{m-1+\gamma} u^{m-1+\gamma} - a_\varepsilon \zeta u^\gamma\right)$$

and

$$\dot{\boldsymbol{\mu}} \in \partial \hat{\mathcal{R}}^{(2)*}_\varepsilon\big(\boldsymbol{\mu}, -\mathrm{D}\hat{\mathcal{E}}^{(2)}_\varepsilon(\boldsymbol{\mu}) - \zeta\big) \iff \dot{u} = \partial_x\big(a_\varepsilon u^m - a_\varepsilon \zeta u^m\big).$$

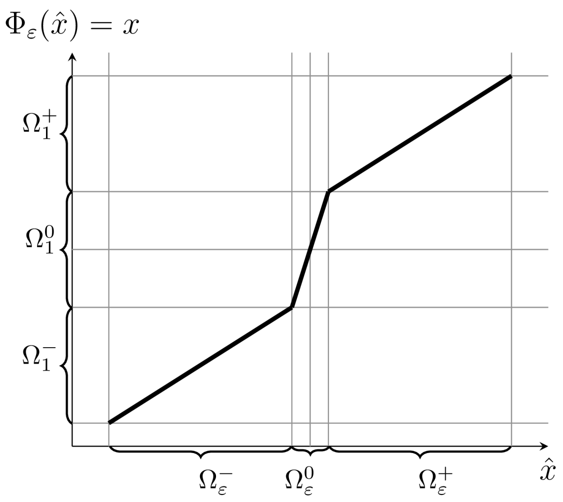

Figure 5.1: Depiction of the graph of the scaling function Φ_ε.

In both cases we pass to the limit in a rescaled gradient system with a fixed domain $\overline{\Omega}_1$. The domain is rescaled via $\Phi_\varepsilon : \overline{\Omega}_\varepsilon \to \overline{\Omega}_1$ with $\Phi(\Omega_\varepsilon^\iota) = \Omega_1^\iota$ for $\iota \in \{-, 0, +\}$ where $\Omega_\varepsilon^0 = (\varepsilon x_-, \varepsilon x_+)$ and $\Omega_\varepsilon^\pm = \{\hat{x} \in \Omega_\varepsilon : \pm\hat{x} > \sup \pm\Omega_\varepsilon^0\}$ (see Figure 5.1). More precisely, the gradient systems are then transformed via the push-forward measure $\mu = (\Phi_\varepsilon)_\# \boldsymbol{\mu}$.

The coefficient $\mathbf{a}_\varepsilon$ is assumed to be of the form

$$\mathbf{a}_\varepsilon(\hat{x}) = \begin{cases} a \circ \Phi_\varepsilon(\hat{x}) & \text{if } \hat{x} \in \overline{\Omega}_1 \setminus \Omega_\varepsilon^0, \\ \varepsilon a \circ \Phi_\varepsilon(\hat{x}) & \text{if } \hat{x} \in \Omega_\varepsilon^0. \end{cases}$$

We assume that a is elliptic, i.e.,

$$0 < \underline{a} \leq a \leq \overline{a} < \infty.$$

In the effective gradient structure it enters the harmonic mean on the membrane layer which is denoted by

$$\operatorname{harm}_{\Omega_1^0}(a) = \left(\int_{\Omega_1^0} \frac{1}{a} \,\mathrm{d}x \right)^{-1}. \tag{5.2}$$

Both gradient systems have the common feature that the measure μ_ε vanishes on the membrane part, i.e., $\mu_\varepsilon(\Omega_1^0) \to 0$ but the density u_ε with respect to $\pi_\varepsilon = (\Phi_\varepsilon)_\# \boldsymbol{\pi}_\varepsilon = \frac{1}{2+\varepsilon} \mathbf{m}_\varepsilon \mathscr{L}^1_{|\Omega_1}$ does not. Here we denote

$$\mathbf{m}_\varepsilon = \begin{cases} 1 & \text{on } \Omega_1 \setminus \Omega_1^0 \\ \varepsilon & \text{on } \Omega_1. \end{cases} \tag{5.3}$$

Hence, the limits of the dissipative parts

$$\mathfrak{D}_\varepsilon^{\mathrm{dual}}(\mu) = \int_0^T \mathcal{R}_\varepsilon^*\big(\mu, -\mathrm{D}\mathcal{E}_\varepsilon(\mu)\big) \,\mathrm{d}t \tag{5.4a}$$

and

$$\mathfrak{D}_\varepsilon^{\mathrm{prim}}(\dot\mu) = \int_0^T \mathcal{R}_\varepsilon(\mu,\dot\mu)\,\mathrm{d}t \tag{5.4b}$$

depend on the limiting density in the membrane part which cannot be related to the limiting measure. Minimizing the sum of the limiting dissipative parts leads then to the effective dissipation potential.

5.1 The Tsallis-Wasserstein setting

In the following we analyze the gradient system (5.1a). The energy here is given by the Tsallis entropy E_m proposed by Constantino Tsallis in [Tsa88]. As shown in [Ott01] the porous medium equation is a Wasserstein flow with respect to the Tsallis entropy. In the following we restrict ourselves to the case $\mathsf{m}(u) = u^\gamma$ with $0 < \gamma \le 1$.

The effective dissipation potential $\mathcal{R}_{\text{eff}}$ is composed of two parts, a diffusion part $\mathcal{R}_{\text{bulk}}$ on the bulk part $\Omega_1 \setminus \overline{\Omega}_1^0$ and a jump part $\mathcal{R}_{\text{memb}}$ on the membrane part Ω_1^0. Moreover, we give in (5.13) and (5.14) a primal and dual characterization for $\mathcal{R}_{\text{memb}}$. Namely,

$$\begin{aligned}
&\mathcal{R}^\zeta_{\text{memb}}(\mu,\dot\mu) + \mathcal{R}^{\zeta*}_{\text{memb}}\big(\mu, [\![-\mathrm{E}_m'(\mu)+\zeta]\!]\big)\\
&= \inf_u \frac{1}{2}\int_{\Omega_1^0} |\partial_x(\mathrm{E}_m'(u)-\zeta)|^2\mathsf{m}(u) + \frac{\kappa^2}{\mathsf{m}(u)}\,\mathrm{d}z\\
&= \sup_\xi \left\{ \langle \xi,\kappa\rangle + \frac{1}{2}\inf_u \left\{ \int_{\Omega_1^0} |\partial_x(\mathrm{E}_m'(u)-\zeta)|^2\mathsf{m}(u)\,\mathrm{d}z - h_u\xi^2 \right\}\right\},
\end{aligned}$$

where $h_u := \mathrm{harm}_{\Omega_1^0}(\mathsf{m}(u))$ and $\dot\mu = \kappa[\![\cdot]\!]$. We have the contact relation

$$\mathcal{R}_{\text{memb}}(\mu,\dot\mu) + \mathcal{R}^*_{\text{memb}}\big(\mu, -[\![\mathrm{E}_m'(u)]\!]\big) = -\kappa[\![\mathrm{E}_m'(u)]\!] \iff \kappa = -\int_{\gamma_- u^-}^{\gamma_+ u_+} \mathrm{E}_m''(u)\mathsf{m}(u)\,\mathrm{d}u,$$

which will be interpreted as a kinetic relation for jumps through the membrane.

To study the limiting procedure we write the transformed gradient system on $\overline{\Omega}_1$ explicitly. Note that $\Phi_\varepsilon' = \varepsilon^{-1}$ on Ω_ε^0 and $\Phi_\varepsilon' = 1$ on $\Omega_\varepsilon \setminus \Omega_\varepsilon^0$. For $\boldsymbol{\mu} \in \mathcal{P}(\overline{\Omega}_\varepsilon)$ with $\mathrm{d}\boldsymbol{\mu} = \mathbf{u}\,\mathrm{d}\boldsymbol{\pi}_\varepsilon$ we compute the push-forward $\mu = (\Phi_\varepsilon)_\#\boldsymbol{\mu}$ and obtain $\mathrm{d}\mu = \mathbf{m}_\varepsilon u\,\mathrm{d}\pi_\varepsilon$, where $u = \mathbf{u}\circ\Phi_\varepsilon$ with $\mathbf{m}_\varepsilon$ given in (5.3). Moreover, the normalized Lebesgue measure π_ε transforms to $\pi_\varepsilon = (\Phi_\varepsilon)_\#\boldsymbol{\pi}_\varepsilon$ with $\mathrm{d}\pi_\varepsilon = c_\varepsilon\mathbf{m}_\varepsilon\,\mathrm{d}x$ with $c_\varepsilon = 1/(2+\varepsilon)$. Hence, the transformed gradient system reads

$$\begin{aligned}
\mathrm{X} &= \mathcal{P}(\overline{\Omega}_1),\\
\mathcal{E}_\varepsilon(\mu) &= \begin{cases} \int_{\Omega_1} \mathrm{E}_m(u)\,\mathrm{d}\pi_\varepsilon & \text{if } \mu = u\pi_\varepsilon,\\ \infty & \text{else,}\end{cases},\\
\mathcal{R}^*_\varepsilon(\mu,\xi) &= \frac{1}{2}\int_{\Omega_1} \mathrm{a}|\partial_x\xi|^2\mathsf{m}(u)\,\mathrm{d}\Pi_\varepsilon,
\end{aligned}$$

where we used the relation $\mathbf{a}_\varepsilon = \mathbf{m}_\varepsilon \mathrm{a} \circ \Phi_\varepsilon$ and denote $\Pi_\varepsilon = c_\varepsilon \mathscr{L}^1_{|\Omega_1}$. Note that Π_ε has no special scaling in the membrane part but $\pi_\varepsilon = c_\varepsilon \mathbf{m}_\varepsilon \mathscr{L}^1_{|\Omega_1}$ has.

For the limit passage, we have to cope with the non-linearity $\rho = \mathsf{m}(u)$ and the fact that $\mu_\varepsilon(\Omega_1^0) \to 0$ but $u_{\varepsilon|_{\Omega_1^0}} \not\to 0$. To get rid of the dependence $u_{|_{\Omega_1^0}}$ we minimize over all possible shapes u with fixed boundary conditions. Moreover, we restrict our analysis to the case where $(u, B) \mapsto u^\gamma \big(\mathrm{E}''_m(u)\big)^2 B^2$ is convex.

5.1.1 Compactness

We treat the nonlinearity by applying an Aubin-Lion Lemma which yields strong convergence on the bulk part $\Omega_1 \setminus \Omega_1^0$. Hence, we are concerned with obtaining suitable a priori bounds.

Lemma 5.1. *Let $0 < \gamma \le 1 < m$ and $2 \le m + \gamma$. Let $\mu_\varepsilon = u_\varepsilon \pi_\varepsilon$ be such that*

$$\sup_\varepsilon \big\{ \mathfrak{D}^{\mathrm{dual}}_\varepsilon(\mu_\varepsilon) + \sup_{t\in[0,T]} \mathcal{E}_\varepsilon\big(\mu_\varepsilon(t)\big) \big\} < \infty.$$

Then

$$\sup_\varepsilon \int_0^T \int_{\Omega_1} |\partial_x u_\varepsilon^m| + u_\varepsilon^m \,\mathrm{d}x\,\mathrm{d}t < \infty.$$

Proof. We exploit the estimate (2.2) and obtain

$$2\int_{\Omega_1^0} |u^m| \,\mathrm{d}x \le 2\int_{\Omega_1} |\partial_x u^m| \,\mathrm{d}x + \int_{\Omega_1\setminus\Omega_1^0} |u^m| \,\mathrm{d}x. \tag{5.5}$$

Using Hölder's estimate with $v = u^{2-\gamma}$

$$\begin{aligned}\int_{\Omega_1} |\partial_x u^m| \,\mathrm{d}x &\le \Big(\int_{\Omega_1} v \,\mathrm{d}x\Big)^{\frac12} \Big(\int_{\Omega_1} |\partial_x u^m|^2 \frac{1}{v} \,\mathrm{d}x\Big)^{\frac12} \\ &\le c_m^{(1)} \Big(\int_{\Omega_1} v \,\mathrm{d}x\Big)^{\frac12} \Big(\int_{\Omega_1} |\partial_x \mathrm{E}'_m(u)|^2 u^\gamma \,\mathrm{d}x\Big)^{\frac12} \\ &\overset{\text{(i)}}{\le} c_{m,\gamma}^{(2)} \Big(\int_{\Omega_1} u^m \,\mathrm{d}x\Big)^{\frac{2-\gamma}{2m}} \Big(\int_{\Omega_1} |\partial_x \mathrm{E}'_m(u)|^2 u^\gamma \,\mathrm{d}x\Big)^{\frac12}\end{aligned}$$

with $c_m^{(1)} = m^{-1}$ and $c_{m,\gamma}^{(2)} = m^{-1}|\Omega_1|^{\frac{m+\gamma-2}{2m}}$. In (i) we used that $1 \le \frac{m}{2-\gamma}$ by assumption. Applying twice Young's estimate $ab \le \frac1p a^p + \frac1q b^q$ with conjugate exponents p and q we obtain with $p_1 = \frac12$ and $p_2 = \frac{m}{2-\gamma}$ that

$$\int_{\Omega_1} |\partial_x u^m| \le \frac{1}{2p_2}\int_{\Omega_1} u^m \,\mathrm{d}x + \frac{1}{2q_2} + \frac{(c_{m,\gamma}^{(2)})^2}{2}\int_{\Omega_1} |\partial_x \mathrm{E}'_m(u)|^2 u^\gamma \,\mathrm{d}x.$$

Inserting this into (5.5) and using $c_1 = \frac{1}{2p_2} < 1$ we conclude

$$2(1-c_1)\int_{\Omega_1^0} |u^m| \,\mathrm{d}x \le \frac{1}{q_2} + (1+p_2^{-1})\int_{\Omega_1\setminus\Omega_1^0} |u^m| \,\mathrm{d}x + (c_{m,\gamma}^{(2)})^2 \int_{\Omega_1} |\partial_x \mathrm{E}'_m(u)|^2 u^\gamma \,\mathrm{d}x. \tag{5.6}$$

The bound on the energies $\sup_{\varepsilon,t} \int_{\Omega_1\setminus\Omega_1^0} u_\varepsilon^m(t)\,\mathrm{d}x < \infty$ controls the second term on the right-hand side, while the bound on $\mathfrak{D}_\varepsilon^{\mathrm{dual}}(\mu_\varepsilon)$ controls the time integral of the third term. Thus the mass of u^m is bounded on the whole time-space cylinder $(0,T)\times\Omega_1$. Inserting this into (5.6) gives the final result. $\square$

The bound on u^m is used in the following Lemma to prove boundedness of $\dot{\mu}_\varepsilon$.

Lemma 5.2. *Let* $0<\gamma\le 1<m$ *and* $2\le m+\gamma$. *Let* μ_ε *such that*

$$\sup_\varepsilon \big\{ \mathfrak{D}_\varepsilon(\mu_\varepsilon) + \sup_{t\in[0,T]} \mathcal{E}_\varepsilon\big(\mu_\varepsilon(t)\big)\big\} < \infty.$$

Then $\dot{\mu}_\varepsilon$ *is bounded in* $\mathrm{L}^q\big(0,T;\mathrm{W}^{1,p}(\Omega_1)^*\big)$ *where* $p=\frac{2m}{m-\gamma}$ *and* q *is conjugate to* p.

Proof. The crucial point is an estimate for the dual part $\mathcal{R}_\varepsilon^*(\mu_\varepsilon,\xi)$. To do so, we use Hölder's estimate to obtain

$$\int_0^T \mathcal{R}^*(\mu_\varepsilon,\xi)\,\mathrm{d}t \le \overline{a}\left(\int_0^T\int_{\Omega_1} u^m\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{\gamma}{m}} \left(\int_0^T\int_{\Omega_1} |\partial_x\xi|^{2p_1}\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{p_1}},$$

where $p_1=\frac{m}{m-\gamma}$ satisfying $\frac{\gamma}{m}+\frac{1}{p_1}=1$. Note that $2p_1=p$. Using the Fenchel-Young estimate

$$\langle\xi,\dot{\mu}_\varepsilon\rangle \le \mathcal{R}_\varepsilon(\mu_\varepsilon,\dot{\mu}_\varepsilon) + \mathcal{R}^*(\mu_\varepsilon,\xi)$$

and Lemma 5.1, which bounds the mass of u^m, we find a constant $C>0$ depend on the mass of u^m and $\mathcal{D}_\varepsilon(\mu_\varepsilon)$ such that

$$\int_0^T \langle\xi,\dot{\mu}_\varepsilon\rangle\,\mathrm{d}t \le C\big(1+\|\partial_x\xi\|^2_{\mathrm{L}^p((0,T)\times\Omega_1)}\big).$$

Taking the supremum over all ξ such that $\|\partial_x\xi\|_{\mathrm{L}^p((0,T)\times\Omega_1)}\le 1$ we obtain that $\dot{\mu}_\varepsilon$ is bounded in $\mathrm{L}^p\big(0,T;\mathrm{W}^{1,p}(\Omega_1)\big)^* = \mathrm{L}^q\big(0,T;\mathrm{W}^{1,p}(\Omega_1)^*\big)$ ([Hyt16, Thm 1.3.10]). $\square$

The following compactness result on $\partial_x u_\varepsilon$ and $\partial_x\rho_\varepsilon$, where $\rho_\varepsilon=\mathsf{m}(u_\varepsilon)=u_\varepsilon^\gamma$, is used to identify the limits $\lim\rho_\varepsilon=\mathsf{m}(\lim u_\varepsilon)$ via a strong convergence result.

Lemma 5.3. *Let* $0<\gamma\le 1<m$ *and* $2\le m+\gamma$ *and* μ_ε *such that*

$$\sup_\varepsilon \big\{ \mathfrak{D}_\varepsilon(\mu_\varepsilon) + \sup_{t\in[0,T]} \mathcal{E}_\varepsilon\big(\mu_\varepsilon(t)\big)\big\} < \infty.$$

We define $p_1=\frac{2m}{4-m-\gamma}$ *and* $p_\gamma=\frac{2m}{2-m+\gamma}$.

(i) *If additionally* $2m+\gamma\le 4$ *then* $\partial_x u_\varepsilon$ *is bounded in* $\mathrm{L}^{p_1}\big((0,T)\times\Omega_1\big)$.

(ii) *If additionally* $2(m-1)\le\gamma$ *then* $\partial_x u_\varepsilon^\gamma$ *is bounded in* $\mathrm{L}^{p_\gamma}\big((0,T)\times\Omega_1\big)$.

Proof. We apply estimate (2.4) using that u_ε^m is bounded in $\mathrm{L}^1\big((0,T)\times\Omega_1\big)$ and that by the bound on the dissipation we have

$$\sup_\varepsilon \int_0^T \int_{\Omega_1} |\partial_x u_\varepsilon^{m-1+\frac{\gamma-\alpha}{2}}|^2 u_\varepsilon^m \,\mathrm{d}x\,\mathrm{d}t < \infty.$$

Hence, for $1 \le p \le 2$ we have that $|\partial_x u_\varepsilon^{m-1+\frac{\gamma-m}{2}}|^p u_\varepsilon^m = c|\partial_x u_\varepsilon^{m-1+\frac{\gamma}{2}+\frac{2-p}{2p}m}|^p$ is bounded in $\mathrm{L}^1\big((0,T)\times\Omega_1\big)$.

Note that p_1 solves $m-1+\frac{\gamma}{2}+\frac{2-p}{2p}m = 1$ and p_γ solves $m-1+\frac{\gamma}{2}+\frac{2-p}{2p}m=\gamma$.

If $2m+\gamma\le 4$ then $p_1 \le 2$. Since $2 \le m+\gamma$ we have $p_1 \ge m$. This proves (i). If $2(m-1)\le\gamma$ then $p_\gamma \le 2$. Since $2\le m+\gamma$ we have $p_\gamma \ge \frac{m}{\gamma}$. This proves (ii). □

The assumption $2(m-1)\le\gamma$ implies immediately that $m\le\frac{3}{2}$. Note that the set $\Theta := \{(m,\gamma) : \gamma \le 1 < m,\ 2\le m+\gamma,\ 2(m-1)\le\gamma\}$ is not empty and that $(m,\gamma)\in\Theta \Rightarrow \gamma \le 4-2m$ (cf. Figure 5.2). The following lemma is a consequence

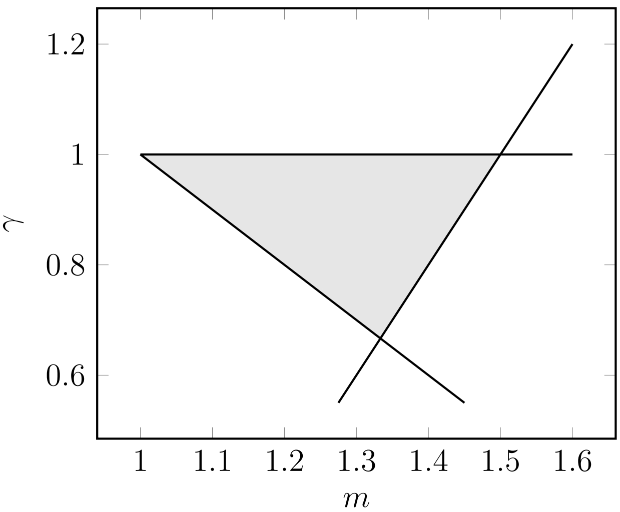

Figure 5.2: Admissible area for (m,γ)

of the a priori bounds.

Lemma 5.4. *Let $(m,\gamma)\in\Theta$. And let μ_ε such that*

$$\sup_\varepsilon \big\{\mathfrak{D}_\varepsilon(\mu_\varepsilon) + \sup_{t\in[0,T]} \mathcal{E}_\varepsilon\big(\mu_\varepsilon(t)\big)\big\} < \infty.$$

Then there exists a subsequence such that $u_\varepsilon|_{\Omega_1^\pm} \to u$ in $\mathrm{L}^q\big(0,T;\mathrm{L}^m(\Omega_1^\pm)\big)$ and $\mu_\varepsilon \to \mu$ in $\mathrm{C}^0\big(0,T;\big(\mathrm{W}^{1,p}(\Omega_1)\big)^*\big)$ where $p=\frac{2m}{m-\gamma}$ and $q\le m$ is conjugate to p.

Proof. Lemma 5.3 gives that μ_ε is bounded in $\mathrm{L}^m\big(0,T;\mathrm{W}^{1,m}(\Omega_1)\big)$ and Lemma 5.2 gives that on $\dot\mu_\varepsilon$ in $\mathrm{L}^q\big(0,T;\mathrm{W}^{1,p}(\Omega_1)^*\big)$. Applying the Aubin-Lion-Lemma [Sim87, Cor. 4] with $\mathrm{W}^{1,m}(\Omega_1) = X \overset{c}{\hookrightarrow} B = \mathrm{L}^m(\Omega_1) \hookrightarrow Y = \mathrm{W}^{1,p}(\Omega_1)^*$ we obtain strong compactness in $\mathrm{L}^q\big(0,T;\mathrm{L}^m(\Omega_1)\big)$. In particular, $\mu_{\varepsilon|_{\Omega_1^\pm}} = u_{\varepsilon|_{\Omega_1^\pm}}$ converges strongly in $\mathrm{L}^q\big(0,T;\mathrm{L}^m(\Omega_1^\pm)\big)$. Additionally, by the energy bound we obtain that μ_ε is bounded in $\mathrm{L}^\infty\big(0,T;\mathrm{L}^m(\Omega_1)\big)$. Applying the Aubin-Lion-Lemma [Sim87, Cor. 4] again we obtain strong compactness in $\mathrm{C}^0\big(0,T;\mathrm{W}^{1,p}(\Omega_1)^*\big)$ by the compact embedding $\mathrm{L}^m(\Omega_1) \overset{c}{\hookrightarrow} \mathrm{W}^{1,p}(\Omega_1)^*$. $\square$

Note that Lemma 5.3 results in $u_\varepsilon^\gamma \rightharpoonup \rho$ in $\mathrm{L}^{m/\gamma}\big(0,T;\mathrm{W}^{1,m/\gamma}(\Omega_1)\big)$. Hence, we find $\rho_{|_{\Omega_1^\pm}} = u^\gamma_{|_{\Omega_1^\pm}}$.

As a direct consequence of Lemma 5.4 and Lemma 5.3 we obtain weak convergence of $\partial_x \mathrm{E}'_m(u_\varepsilon)u_\varepsilon^\gamma$.

Corollary 5.5. *Let $(m,\gamma) \in \Theta$ and μ_ε such that*

$$\sup_\varepsilon \big\{\mathfrak{D}_\varepsilon(\mu_\varepsilon) + \sup_{t\in[0,T]} \mathcal{E}_\varepsilon\big(\mu_\varepsilon(t)\big)\big\} < \infty.$$

Then

$$\partial_x \mathrm{E}'_m(u_\varepsilon)u_\varepsilon^\gamma \rightharpoonup \partial_x \mathrm{E}'_m(u)u^\gamma \quad \textit{in } \mathrm{L}^1\big((0,T)\times\Omega_1\setminus\Omega_1^0\big).$$

Proof. Note that $q = \frac{2m}{m+\gamma} \le m$, hence, $u_\varepsilon \to u$ in $\mathrm{L}^q\big((0,T)\times(\Omega_1\setminus\Omega_1^0)\big)$ by Lemma 5.4. Since u_ε is bounded in $\mathrm{L}^m\big((0,T)\times\Omega_1\setminus\Omega_1^0\big)$ we obtain also strong convergence in $\mathrm{L}^{q'}\big((0,T)\times(\Omega_1\setminus\Omega_1^0)\big)$ for all $q' < m$ if $q < m$. Moreover, $\partial_x u_\varepsilon \rightharpoonup \partial_x u$ in $\mathrm{L}^{p_1}\big((0,T)\times(\Omega_1\setminus\Omega_1^0)\big)$. Using $\partial_x \mathrm{E}'_m(u)\, u^\gamma = c\partial_x u\, u^{m+\gamma-2}$ for some $c>0$ it remains to show that $u_\varepsilon^{m+\gamma-2} \to u^{m+\gamma-2}$ in $\mathrm{L}^{q_1}\big((0,T)\times\Omega_1\setminus\Omega_1^0\big)$, i.e.,

$$2m\frac{m+\gamma-2}{3m+\gamma-4} = (m+\gamma-2)q_1 < m.$$

Since $\gamma < m$ we immediately obtain $m\frac{2m+2\gamma-4}{3m+\gamma-4} < m\frac{3m+\gamma-4}{3m+\gamma-4} = m$. $\square$

For $(m,\gamma)\in\Theta$ we prove the validity of the chain-rule.

Lemma 5.6. *Let $(m,\gamma)\in\Theta$ and $\mathfrak{D}_\varepsilon(\mu) < \infty$. Then for any $\varepsilon > 0$ we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_\varepsilon(\mu) = \langle \mathrm{D}\mathcal{E}_\varepsilon(\mu), \dot\mu\rangle.$$

Proof. With $F : u \mapsto u^{m-1+\gamma/2}$ we obtain that $F(u) \in \mathrm{L}^2\big(0,T;\mathrm{H}^1(\Omega_1)\big)$ since $2m+\gamma \le 4$. In particular, $u \in \mathrm{L}^{2m-2+\gamma}\big(0,T;\mathrm{C}^0(\Omega_1)\big)$. Note that $1 < 2m-2+\gamma$. With the solution v to the kinetic relation

$$\langle\varphi,\dot\mu\rangle = \int_{\Omega_1} a\mathbf{v}\partial_x\varphi\; u^\gamma \,\mathrm{d}x$$

we estimate

$$|\langle\varphi,\dot\mu\rangle| \le \|\mathbf{v}\|_{\mathrm{L}^2(\mathsf{m}(u)\,\mathrm{d}x,\Omega_1)}\|u\|_\infty^{\gamma/2}\|\partial_x\varphi\|_{\mathrm{L}^2(\Omega_1)},$$

i.e., $\dot\mu \in \mathrm{L}^2\big(0,T;\mathrm{H}^1(\Omega_1)^*\big)$.

For the case $u \geq \delta > 0$ we also obtain that $\partial_x \mathrm{E}_m(u) \in \mathrm{L}^2\big(0,T;\mathrm{H}^1(\Omega_1)\big)$. Using [MRS13, Prop. 2.4] we conclude

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_\varepsilon(\mu) = \langle \mathrm{D}\mathcal{E}_\varepsilon(\mu), \dot\mu\rangle.$$

For the general case $u \geq 0$, we define $u_\delta = u + \delta$. We check that $\mathfrak{D}_\varepsilon(\mu_\delta) \leq \mathfrak{D}_\varepsilon(\mu) < \infty$. We easily see that $\mathbf{v}_\delta = \mathbf{v}\frac{\mathsf{m}(u)}{\mathsf{m}(u_\delta)}$ and $|\mathbf{v}_\delta|^2\mathsf{m}(u_\delta) \leq |\mathbf{v}|^2\mathsf{m}(u)$ since m is monotone. Since $2(m-2)+\gamma \leq 0$ we have that $u \mapsto |\mathrm{E}''_m(u)|^2\mathsf{m}(u)$ is monotonously decreasing. Moreover, $\mathrm{E}''_m(u_\delta) \leq \mathrm{E}''_m(u)$ since $m < 2$. Thus, $|\mathbf{v}_\delta\partial_x\mathrm{E}'_m(u_\delta)\mathsf{m}(u_\delta)| \leq |\mathbf{v}\partial_x\mathrm{E}'_m(u)\mathsf{m}(u)|$. By dominated convergence we pass to the limit in

$$\int_{t_1}^{t_2} \langle \mathrm{D}\mathcal{E}_\varepsilon(\mu_\delta), \dot\mu_\delta\rangle \,\mathrm{d}t$$

and obtain the result. □

5.1.2 The Γ-liminf estimate of $\mathfrak{D}_\varepsilon$

In the sequel, we are concerned with the Γ-liminf estimate for

$$\mathfrak{D}_\varepsilon^\zeta : \mu \mapsto \int_0^T \mathcal{R}_\varepsilon(\mu,\dot\mu) + \mathcal{R}_\varepsilon^*\big(\mu, -\mathrm{D}\mathcal{E}_\varepsilon(\mu)+\zeta\big)\,\mathrm{d}t,$$

where $\zeta \in \mathrm{W}^{1,\infty}(\Omega_1)$. We recall important objects for the limits passage:

(i) The continuity equation

$$\langle \xi, \dot\mu\rangle = \int_{\Omega_1} \mathrm{a}(x)\partial_x\xi\ \mathbf{v}\ u^\gamma \,\mathrm{d}\pi_\varepsilon(x) \tag{5.7}$$

for all $\xi \in \mathrm{C}^1(\Omega_1)$.

(ii) The primal dissipation potential

$$\mathcal{R}_\varepsilon(\mu,\dot\mu) = \frac{1}{2}\int_{\Omega_1} \mathrm{a}(x)|\mathbf{v}|^2 u^\gamma \,\mathrm{d}\pi_\varepsilon(x) \tag{5.8}$$

where $\mathbf{v}$ satisfies the continuity equation, and the slope term

$$\mathcal{R}_\varepsilon^*(\mu, -\mathrm{D}\mathcal{E}_\varepsilon(u)+\zeta) = \frac{1}{2}\int_{\Omega_1} \mathrm{a}(x)|\partial_x(\mathrm{E}'_m(u)-\zeta)|^2 u^\gamma \,\mathrm{d}\pi_\varepsilon(x).$$

For $(m,\gamma) \in \Theta$ we have convexity of $(a,b) \mapsto a^2 b^{2m+\gamma-4}$ since $2m+\gamma \leq 4$ and convexity of $(a,b) \mapsto a^2 b^{\frac{2(m-1)}{\gamma}-1}$ since $2(m-1) \leq \gamma$. We combine the convexity with the weak convergence stated in Corollary 5.5 and the strong convergence stated in Lemma 5.4. We obtain the liminf estimate for the dual part

$$\mathfrak{D}_\varepsilon^{\zeta,\mathrm{dual}}(\mu) = \int_0^T \mathcal{R}_\varepsilon^*(\mu, -\mathrm{D}\mathcal{E}_\varepsilon(u)+\zeta)\,\mathrm{d}t.$$

More precisely, Lemma 5.4 gives only strong convergence on the bulk part $\Omega_1 \setminus \Omega_1^0$. On the membrane part Ω_1^0 we use a convex envelope. Details are given in Proposition 5.8 and Proposition 5.9 below. For the primal part

$$\mathfrak{D}_\varepsilon^{\mathrm{prim}}(\mu) = \int_0^T \mathcal{R}_\varepsilon(\mu, \dot{\mu})\,\mathrm{d}t$$

we use [AGS05, Thm 5.4.4] (see Section 2.2) to pass to the limit using the representation (5.8) of $\mathcal{R}_\varepsilon$ via the continuity equation.

Moreover, we use that $\mu_\varepsilon(\Omega_1^0) \to 0$ and obtain for all $\eta \in \mathrm{C}_c^1(\Omega_1^0)$ that

$$0 = \lim_{\varepsilon \downarrow 0} \langle \eta, \dot{\mu}_\varepsilon \rangle = \int_{\Omega_1^0} \mathrm{a} \partial_x \eta \mathbf{v}_0 u^\gamma \,\mathrm{d}\pi_0(x).$$

In other words, $x \mapsto \mathrm{a}\mathbf{v}_0 u^\gamma \equiv \mathrm{harm}_{\Omega_1^0}(\mathrm{a})\,\kappa \in \mathbb{R}$ is spatially constant. This leads to a dependence of the continuity equation on the jump $[\![\eta]\!] := \eta(x_+) - \eta(x_-)$.

Since $\dot{\mu}_{|_{\Omega_1^0}} = 0$ the evolution in the membrane is determined by the boundary values $u_\pm := u(x_\pm)$. This is reflected by the minimization problem

$$\mathcal{M}(u_\pm, \kappa) := \inf_w \int_{\Omega_1^0} |\partial_x \mathrm{E}_m'(w)|^2 w^\gamma + \frac{\kappa^2}{w^\gamma}\,\mathrm{d}x$$

with w satisfying $w(x_\pm) = u_\pm$ in the case $\zeta \equiv 0$.

First, we estimate the contribution on the bulk $\Omega_1 \setminus \Omega_1^0$ for the primal and dual dissipation separately. The second step is to estimate the contribution of the dissipation in the membrane which lacks convexity, i.e.,

$$(\rho, g) \mapsto \tilde{\mathcal{K}}(\rho, w; \zeta) = \frac{1}{2} \int_{\Omega_1^0} |\partial_x \mathrm{E}_m'(\rho^{1/\gamma}) - \partial_x \zeta|^2 \mathrm{a}\rho + \frac{|g|^2}{\mathrm{a}\rho}\,\mathrm{d}\Pi_0 \tag{5.9}$$

is not convex in the case $2 \lneqq m + \gamma$ and $\zeta \neq 0$. Here g is a place holder for $a\mathbf{v}u^\gamma$. This leads to the minimization problem

$$\tilde{\mathcal{M}}(u_\pm, \kappa; \zeta) := \inf_\rho \tilde{\mathcal{K}}^{**}(\rho, \kappa; \zeta) \tag{5.10}$$

subject to the boundary conditions $\rho_\pm = u_\pm^\gamma$, where $\tilde{\mathcal{K}}^{**}$ is the convex envelope of $\tilde{\mathcal{K}}$ with respect to the variable (ρ, g). We start with the primal (bulk) part.

Proposition 5.7. *Let $(m, \gamma) \in \Theta$ and $\{\mu_\varepsilon\}_\varepsilon \subset \mathcal{P}(\overline{\Omega}_1)$ such that the densities with respect to the measure π_ε converge weakly, i.e., $u_\varepsilon \rightharpoonup u$ in $\mathrm{L}^m\big(0,T;\mathrm{W}^{1,m}(\Omega_1)\big)$ and*

$$\sup_\varepsilon \left\{ \mathfrak{D}_\varepsilon(\mu_\varepsilon) + \sup_{t \in [0,T]} \mathcal{E}_\varepsilon\big(\mu_\varepsilon(t)\big) \right\} < \infty.$$

Then on the bulk (excluding the membrane) we obtain

$$\liminf_{\varepsilon \downarrow 0} \int_0^T \int_{\Omega_1 \setminus \Omega_1^0} |\mathbf{v}_\varepsilon|^2 \mathrm{a}\rho_\varepsilon \,\mathrm{d}x\,\mathrm{d}t \geq \int_0^T \int_{\Omega_1 \setminus \Omega_1^0} |\mathbf{v}|^2 \mathrm{a}\rho \,\mathrm{d}x$$

with $\mathbf{v} \in \mathrm{L}^2\big(\rho \,\mathrm{d}x\,\mathrm{d}t, (0,T)\times\Omega_1\setminus\Omega_1^0\big)$ *and* $\tilde\kappa \in \mathrm{L}^2\big((\int(\mathrm{a}\rho)^{-1}\,\mathrm{d}x)\,\mathrm{d}t,(0,T)\big)$ *satisfying*

$$\langle\dot\mu,\varphi\rangle = c_0\int_{\Omega_1\setminus\Omega_1^0}\partial_x\varphi\cdot\mathbf{v}\ \mathrm{a}\rho\,\mathrm{d}x + c_0\tilde\kappa[\![\varphi]\!],\qquad [\![\varphi]\!] = \varphi^+-\varphi^-.$$

Moreover, $\tilde\kappa$ *is the weak* $\mathrm{L}^1\big((0,T)\times\Omega_1^0\big)$*-limit of* $\mathrm{a}\mathbf{v}_\varepsilon\rho_\varepsilon$*.*

Proof. Applying [AGS05, Thm 5.4.4] (see Section 2.2) we obtain a weak limit $\mathbf{v}\in\mathrm{L}^2\big(\rho\,\mathrm{d}x\,\mathrm{d}t,(0,T)\times\Omega_1\big)$ such that

$$\liminf_{\varepsilon\downarrow0}\int_0^T\int_{\Omega_1\setminus\Omega_1^0}|\mathbf{v}_\varepsilon|^2\mathrm{a}\rho_\varepsilon\,\mathrm{d}x\,\mathrm{d}t \geq \int_0^T\int_{\Omega_1\setminus\Omega_1^0}|\mathbf{v}|^2\mathrm{a}\rho\,\mathrm{d}x\,\mathrm{d}t.$$

Since $\mathrm{d}\mu_\varepsilon = u_\varepsilon\,\mathrm{d}\pi_\varepsilon = c_\varepsilon\mathbf{m}_\varepsilon u_\varepsilon\,\mathrm{d}x$ with $\mathbf{m}_\varepsilon=\varepsilon$ on Ω_1^0 we obtain $\mu(\Omega_1^0)=0$ and for $\varphi\in\mathrm{C}_0^1\big([0,T]\times\Omega_1^0\big)$ that

$$0=\int_0^T\langle\dot\mu,\varphi\rangle\,\mathrm{d}t = \int_0^T\int_{\Omega_1^0}\partial_x\varphi\cdot\mathbf{v}\ \mathrm{a}\rho\,\mathrm{d}\Pi_0\,\mathrm{d}t,$$

i.e., $\mathbf{v}\ \mathrm{a}\rho\equiv\tilde\kappa\in\mathrm{L}^2\big((\int(\mathrm{a}\rho)^{-1}\,\mathrm{d}x)\,\mathrm{d}t,(0,T)\big)$. This leads to the continuity equation

$$\forall\,\varphi\in\mathrm{C}_0^1\big([0,T]\times\Omega_1\big):\ \int_0^T\langle\dot\mu,\varphi\rangle\,\mathrm{d}t = \int_0^T\int_{\Omega_1\setminus\Omega_1^0}\partial_x\varphi\cdot\mathbf{v}\ \mathrm{a}\rho\,\mathrm{d}\pi_0 + c_0\tilde\kappa[\![\varphi]\!]\,\mathrm{d}t.$$

Note that $\mathrm{a}\mathbf{v}_\varepsilon\rho_\varepsilon$ is equi-integrable, hence, the weak L^1-limit and the weak limit in the sense of [AGS05, Thm. 5.4.4] coincide. □

For the dual (bulk) part we use the strong convergence given by Lemma 5.4 and the weak convergence given by Corollary 5.5.

Proposition 5.8. *Let* $(m,\gamma)\in\Theta$ *and* $\{\mu_\varepsilon\}_\varepsilon\subset\mathcal{P}(\overline{\Omega}_1)$ *such that the densities satisfy* $u_\varepsilon\rightharpoonup u$ *in* $\mathrm{L}^m\big(0,T;\mathrm{W}^{1,m}(\Omega_1)\big)$ *and*

$$\sup_\varepsilon\left\{\mathfrak{D}_\varepsilon(\mu_\varepsilon)+\sup_{t\in[0,T]}\mathcal{E}_\varepsilon\big(\mu_\varepsilon(t)\big)\right\}<\infty.$$

Then on the bulk (excluding the membrane) we have

$$\liminf_{\varepsilon\downarrow0}\int_0^T\int_{\Omega_1\setminus\Omega_1^0}|\partial_x\mathrm{E}_m'(u_\varepsilon)-\partial_x\zeta|^2\mathrm{a}\rho_\varepsilon\,\mathrm{d}x\,\mathrm{d}t\geq\int_0^T\int_{\Omega_1\setminus\Omega_1^0}|\partial_x\mathrm{E}_m'(u)-\partial_x\zeta|^2\mathrm{a}\rho\,\mathrm{d}x.$$

In order to prove the full Γ-liminf estimate it remains to estimate the membrane part of the total dissipation.

Proposition 5.9. *Let* $(m,\gamma)\in\Theta$ *and* $\{\mu_\varepsilon\}_\varepsilon\subset\mathcal{P}(\overline{\Omega}_1)$ *such that the densities satisfy* $u_\varepsilon\rightharpoonup u$ *in* $\mathrm{L}^m\big(0,T;\mathrm{W}^{1,m}(\Omega_1)\big)$ *and*

$$\sup_\varepsilon\left\{\mathfrak{D}_\varepsilon(\mu_\varepsilon)+\sup_{t\in[0,T]}\mathcal{E}_\varepsilon\big(\mu_\varepsilon(t)\big)\right\}<\infty.$$

Then on the membrane we have

$$\liminf_{\varepsilon} \int_0^T \tilde{\mathcal{K}}(\rho_\varepsilon, \mathbf{v}_\varepsilon; \zeta)\,\mathrm{d}t \geq \int_0^T \inf_{\rho} \tilde{\mathcal{K}}^{**}(\rho, \tilde{\kappa}; \zeta)\,\mathrm{d}t,$$

where the infimum is taken subject to the boundary conditions $\rho_\pm = u_\pm^\gamma$ *and* $\tilde{\mathcal{K}}$ *is given in* (5.9).

Proof. By Proposition 5.7 we have $(\rho_\varepsilon, \mathbf{v}_\varepsilon \mathrm{a} \rho_\varepsilon) \rightharpoonup (\rho, \tilde{\kappa})$. Hence, we obtain

$$\begin{aligned}\liminf_{\varepsilon} \int_0^T \tilde{\mathcal{K}}(\rho_\varepsilon, \mathbf{v}_\varepsilon \mathrm{a}\rho_\varepsilon; \zeta)\,\mathrm{d}t &\geq \liminf_{\varepsilon} \int_0^T \tilde{\mathcal{K}}^{**}(\rho_\varepsilon, \mathbf{v}_\varepsilon \mathrm{a}\rho_\varepsilon; \zeta)\,\mathrm{d}t \\ &\overset{\text{(i)}}{\geq} \int_0^T \tilde{\mathcal{K}}^{**}(\rho, \tilde{\kappa}; \zeta)\,\mathrm{d}t \geq \int_0^T \inf_{\rho} \tilde{\mathcal{K}}^{**}(\rho, \tilde{\kappa}; \zeta)\,\mathrm{d}t.\end{aligned}$$

Here we used in (i) that $(\rho, w) \mapsto \tilde{\mathcal{K}}^{**}(\rho, w; \zeta)$ is jointly convex. □

hence, we arrive at the following estimate

$$\liminf_{\varepsilon \downarrow 0} \mathfrak{D}_\varepsilon^\zeta(\mu_\varepsilon) \geq \int_0^T \left\{ \mathcal{R}_{\mathrm{bulk}}(\mu, \dot{\mu}_{\mathbf{v}}) + \mathcal{R}^*_{\mathrm{bulk}}(\mu, \mathrm{E}'_m(u) - \zeta) + \tilde{\mathcal{M}}(u_\pm, \kappa; \zeta) \right\} \mathrm{d}t,$$

with $\tilde{\mathcal{M}}$ given by (5.10) and the decomposition $\dot{\mu} = \dot{\mu}_{\mathbf{v}} + \dot{\mu}_{\tilde{\kappa}}$ where

$$\langle \dot{\mu}_{\mathbf{v}}, \varphi \rangle = \int_{\Omega_1 \setminus \Omega_1^0} \partial_x \varphi \cdot \mathbf{v}\ \mathrm{a}\rho\,\mathrm{d}\pi_0 \quad \text{and} \quad \langle \dot{\mu}_{\tilde{\kappa}}, \varphi \rangle = c_0 \tilde{\kappa} [\![\varphi]\!]. \tag{5.11}$$

The bulk dissipation potential is given by

$$\mathcal{R}_{\mathrm{bulk}}(\mu, \dot{\mu}) = \begin{cases} \int_{\Omega_1 \setminus \Omega_1^0} \frac{1}{2} |\mathbf{v}|^2 \mathrm{a}\rho\,\mathrm{d}\pi_0 & \text{if } \dot{\mu} = \dot{\mu}_{\mathbf{v}} \\ \infty & \text{else.} \end{cases}$$

In the sequel, we investigate properties of $\tilde{\mathcal{K}}^{**}$ in terms of the dependence on the coefficient a and the contact set (1.9). Note that via a change of variables $\mathrm{harm}_{\Omega_1^0}(\mathrm{a})/\mathrm{a}\,\mathrm{d}x = \mathrm{d}z$ and $\tilde{\kappa} = \mathrm{harm}_{\Omega_1^0}(\mathrm{a})\,\kappa$ we obtain

$$\tilde{\mathcal{K}}(\rho, \tilde{\kappa}; \zeta) = \frac{c_0\,\mathrm{harm}_{\Omega_1^0}(\mathrm{a})}{2} \int_{\Omega_1^0} |\partial_x \mathrm{E}'_m(u) - \partial_x \zeta|^2 u^\gamma + \frac{\kappa^2}{u^\gamma}\,\mathrm{d}z =: \mathrm{harm}_{\Omega_1^0}(\mathrm{a})\,\mathcal{K}(\rho, \kappa; \zeta).$$

As a consequence, we have $\tilde{\mathcal{K}}^{**}(\rho, \mathrm{harm}_{\Omega_1^0}(\mathrm{a})\,\kappa; \zeta) = \mathrm{harm}_{\Omega_1^0}(\mathrm{a})\,\mathcal{K}^{**}(\rho, \kappa; \zeta)$. In the remainder of this subsection, we are concerned with the contact set of $\mathcal{K}^{**}$. For all ρ satisfying the boundary condition $\rho_\pm = u_\pm^\gamma$ we find by Young's estimate

$$\mathcal{K}(\rho, \kappa; \zeta) \geq -\kappa [\![\mathrm{E}'_m(u) - \zeta]\!].$$

In particular, since $\mathcal{K}^{**}$ is the supremum over convex functions $\mathcal{F}$ satisfying $\mathcal{K}(\rho, \kappa; \eta) \geq \mathcal{F}(\rho, \kappa)$ we find

$$\mathcal{K}(\rho, \kappa; \zeta) \geq \mathcal{K}^{**}(\rho, \kappa; \zeta) \geq -\kappa [\![\mathrm{E}'_m(u) - \zeta]\!]. \tag{5.12}$$

Note that $\kappa[\![\mathrm{E}'_m(u)-\zeta]\!]$ does not depend on ρ since the boundary conditions are fixed. Moreover, we obtain the optimality condition

$$\begin{aligned}&\mathcal{K}(\rho_0,\kappa;\zeta)=-\kappa[\![\mathrm{E}'_m(\rho^{1/\gamma})-\zeta]\!]\\ \iff\quad &\partial_x\Big(\big(\partial_x\mathrm{E}'_m(\rho_0^{1/\gamma})-\partial_x\zeta\big)\rho_0\Big)=0 \text{ and } \kappa=-\big(\partial_x\mathrm{E}'_m(\rho_0^{1/\gamma})-\partial_x\zeta\big)\rho_0.\end{aligned}$$

Note that $\mathcal{M}(u_\pm,\kappa;\zeta)$ is convex in κ since $\mathcal{K}^{**}$ is jointly convex. Moreover, $\mathcal{M}$ is monotonously increasing in $|\kappa|$, thus $\mathcal{M}$ defines a dissipation potential via

$$\mathcal{R}^\zeta_{\mathrm{memb}}(u_\pm,\dot\mu)=\begin{cases}\mathrm{harm}_{\Omega_1^0}(\mathrm{a})\,\big(\mathcal{M}(u_\pm,\kappa;\zeta)-\mathcal{M}(u_\pm,0;\zeta)\big) & \text{if } \dot\mu=\dot\mu_\kappa,\\ \infty & \text{else.}\end{cases}\tag{5.13}$$

Concerning the contact set we find the following.

Lemma 5.10. *We have the identity*

$$\mathrm{harm}_{\Omega_1^0}(\mathrm{a})\,\mathcal{M}^\zeta(u_\pm,0)=\mathcal{R}^{\zeta*}_{\mathrm{memb}}(u_\pm,-\mathrm{E}'_m(u)+\zeta).$$

Moreover,

$$\begin{aligned}&\mathcal{R}^\zeta_{\mathrm{memb}}(u_\pm,\kappa)+\mathcal{R}^{\zeta*}_{\mathrm{memb}}(u_\pm,-\mathrm{E}'_m(u)+\zeta)=\langle-[\![\mathrm{E}'_m(u)-\zeta]\!],\kappa\rangle\\ &\Leftrightarrow\ \exists\,\rho_0\in\mathrm{H}^1(\Omega_1^0):\ \partial_x\Big(\big(\partial_x\mathrm{E}'_m(\rho_0^{1/\gamma})-\partial_x\zeta\big)\rho_0\Big)=0 \textit{ and } \kappa=-\big(\partial_x\mathrm{E}'_m(\rho_0^{1/\gamma})-\partial_x\zeta\big)\rho_0.\end{aligned}$$

Proof. We compute

$$\begin{aligned}&\mathcal{R}^{\zeta*}_{\mathrm{memb}}(u_\pm,-\mathrm{E}'_m(\mu)+\zeta)-\mathrm{harm}_{\Omega_1^0}(\mathrm{a})\,\mathcal{M}(u_\pm,0;\zeta)\\ &=\sup_\kappa\ \mathrm{harm}_{\Omega_1^0}(\mathrm{a})\,\big(\langle-[\![\mathrm{E}'_m(u)-\zeta]\!],\kappa\rangle-\mathcal{M}(u_\pm,\kappa;\zeta)\big)\\ &=\sup_\kappa\sup_u\ \mathrm{harm}_{\Omega_1^0}(\mathrm{a})\,\big(\langle-[\![\mathrm{E}'_m(u)-\zeta]\!],\kappa\rangle-\mathcal{K}^{**}(u^\gamma,\kappa;\zeta)\big)\end{aligned}$$

Using (5.12) we find

$$\begin{aligned}0&=\sup_\kappa\sup_u\ \langle-[\![\mathrm{E}'_m(u)-\zeta]\!],\kappa\rangle-\mathcal{K}^{**}(u^\gamma,\kappa;\zeta)\\ &\geq\sup_\kappa\sup_u\ \langle-[\![\mathrm{E}'_m(u)-\zeta]\!],\kappa\rangle-\mathcal{K}(u^\gamma,\kappa;\zeta)=0,\end{aligned}$$

where the maximum is attained if and only if $\exists\,\rho_0\in\mathrm{H}^1(\Omega_1^0)$ with $\rho_0(x_\pm)=u_\pm^\gamma$ such that

$$\partial_x\Big(\big(\partial_x\mathrm{E}'_m(\rho_0^{1/\gamma})-\partial_x\zeta\big)\rho_0\Big)=0\quad\text{and}\quad\kappa=-\big(\partial_x\mathrm{E}'_m(\rho_0^{1/\gamma})-\partial_x\zeta\big)\rho_0.$$

Thus,

$$0=\mathcal{R}^{\zeta*}_{\mathrm{memb}}(u_\pm,-\mathrm{E}'_m(\mu)+\zeta)-\mathrm{harm}_{\Omega_1^0}(\mathrm{a})\,\mathcal{M}(u_\pm,0;\zeta).$$

The existence of ρ_0 is studied in C.4. □

Hence, the effective dissipation potential is the inf-convolution of $\mathcal{R}^{\zeta}_{\mathrm{memb}}$ and $\mathcal{R}_{\mathrm{bulk}}$ where

$$\mathcal{R}^*_{\mathrm{bulk}}(\mu,\xi) = \int_{\Omega_1\setminus\Omega_1^0} \mathrm{a}|\partial_x\xi|^2 u^\gamma \,\mathrm{d}\pi_0.$$

Thus, $\mathcal{R}_{\mathrm{eff}}$ reads

$$\begin{aligned}\mathcal{R}^{\zeta}_{\mathrm{eff}}(\mu,\dot\mu) = \big(\mathcal{R}_{\mathrm{bulk}} \star_{\mathrm{inf}} \mathcal{R}^{\zeta}_{\mathrm{memb}}\big)(\mu,\dot\mu)&\\ = \inf\{\mathcal{R}_{\mathrm{bulk}}(\mu,\dot\mu_{\mathbf{v}}) + \mathcal{R}^{\zeta}_{\mathrm{memb}}(u_\pm,\dot\mu_\kappa)|\ \dot\mu = \dot\mu_{\mathbf{v}} + \dot\mu_\kappa\}&.\end{aligned}$$

The decomposition $\dot\mu = \dot\mu_{\mathbf{v}} + \dot\mu_\kappa$ is introduced in (5.11). Note that $\mathcal{R}^{\zeta}_{\mathrm{eff}}(\mu,\cdot)$ is well-defined on $\mathbb{X} = \mathbb{Y}^*$ where $\mathbb{Y} = \|\cdot\|_{\mathbb{Y}}\text{-cl}\big(\mathrm{C}^1(\Omega_1\setminus\Omega_1^0)\big)$ with $\|\xi\|^2_{\mathbb{Y}} = \|\partial_x\xi\|^2_{\mathrm{L}^2_\mu(\Omega_1\setminus\Omega_1^0)} + [\![\xi]\!]^2$. It is well known that $\big(\mathcal{R}_{\mathrm{bulk}} \star_{\mathrm{inf}} \mathcal{R}^{\zeta}_{\mathrm{memb}}\big)^*(\mu,\xi) = \mathcal{R}^*_{\mathrm{bulk}}(\mu,\xi) + \mathcal{R}^{\zeta*}_{\mathrm{memb}}(\mu,\xi)$ (see e.g. [AB86, Roc66]). However, in order to conclude that we also have

$$\mathcal{R}^{\zeta}_{\mathrm{eff}}(\mu,\dot\mu) = \sup_{\xi\in\mathbb{Y}}\big\{\langle\xi,\dot\mu\rangle - \mathcal{R}^*_{\mathrm{bulk}}(\mu,\xi) - \mathcal{R}^{\zeta*}_{\mathrm{memb}}(\mu,\xi)\big\}$$

we need to exploit that $\mathcal{R}^*_{\mathrm{bulk}}(\mu,\cdot)$ and $\mathcal{R}^{\zeta*}_{\mathrm{memb}}(\mu,\cdot)$ are continuous functionals on $\mathbb{Y}$ (see [Roc66]).

Thus, propositions 5.7, 5.8 and 5.9 lead to the following Γ-liminf estimate.

Theorem 5.11. *Let* $(m,\gamma)\in\Theta$ *and* $\{\mu_\varepsilon\}_\varepsilon \subset \mathcal{P}(\overline{\Omega}_1)$ *such that the densities satisfy* $u_\varepsilon \rightharpoonup u$ *in* $\mathrm{L}^m\big(0,T;\mathrm{W}^{1,m}(\Omega_1)\big)$ *and* $u_\varepsilon^m \rightharpoonup \rho$ *in* $\mathrm{L}^1\big((0,T)\times\Omega_1\big)$*. Then*

$$\liminf_{\varepsilon\downarrow 0} \mathfrak{D}^{\zeta}_{\varepsilon}(\mu_\varepsilon;[0,T]) \geq \mathfrak{D}^{\zeta}_{\mathrm{eff}}(\mu;[0,T]),$$

where $\mathfrak{D}^{\zeta}_{\mathrm{eff}}$ *is defined via*

$$\mathfrak{D}^{\zeta}_{\mathrm{eff}}(\mu;[0,T]) = \int_0^T \mathcal{R}^{\zeta}_{\mathrm{eff}}(\mu,\dot\mu) + \mathcal{R}^{\zeta*}_{\mathrm{eff}}\big(\mu,-\mathrm{D}\mathcal{E}_0(\mu)+\zeta\big)\,\mathrm{d}t$$

There is also a direct expression for the dual dissipation potential. In particular, we see that $\mathcal{R}^{\zeta*}_{\mathrm{memb}}$ depends only on the jump $[\![\xi]\!]$.

Lemma 5.12. *Let*

$$\mathcal{M}_*(u_\pm,[\![\xi]\!];\zeta) = \frac{c_0}{2}\inf_w\left\{\int_{\Omega_1^0} |\partial_x\big(\mathrm{E}'(w)-\zeta\big)|^2\mathsf{m}(w)\,\mathrm{d}x - \mathrm{harm}\,(\mathsf{m}(w))\,[\![\xi]\!]^2\right\} \tag{5.14}$$

where the infimum is taken over all w *such that* $w(x_\pm) = u_\pm$*. Then*

$$\mathcal{R}^{\zeta*}_{\mathrm{memb}}(u_\pm,\xi) = \mathrm{harm}_{\Omega_1^0}(\mathrm{a})\,\big(\mathcal{M}_*(u_\pm,0;\zeta) - \mathcal{M}_*(u_\pm,[\![\xi]\!];\zeta)\big).$$

In the sequel, we neglect the dependence of $\mathcal{M}_*$ on $u_\pm$ and ζ and write $\mathcal{M}_*([\![\xi]\!])$ instead of $\mathcal{M}_*(u_\pm,[\![\xi]\!];\zeta)$.

Proof. Without loss of generality, we assume that $\mathrm{harm}_{\Omega_1^0}(\mathrm{a}) = 1$. Denoting $c_u = \mathrm{harm}_{\Omega_1^0}(\mathsf{m}(u))$, a straight forward computation gives

$$\begin{aligned}
\mathcal{R}^{\zeta*}_{\mathrm{memb}}(u_\pm, \xi) &= \sup_{\kappa,u}\left\{\kappa[\![\xi]\!] - \frac{1}{2}\int_{\Omega_1^0} |\partial_x\big(\mathrm{E}'(w)-\zeta\big)|^2\mathsf{m}(u) + \frac{\kappa^2}{\mathsf{m}(u)}\,\mathrm{d}x\right\} + \mathcal{M}(0)\\
=&\mathcal{M}(0) - \frac{1}{2}\inf_{\kappa,u}\left\{\left(\frac{\kappa}{\sqrt{c_u}} - \sqrt{c_u}[\![\xi]\!]\right)^2 + \int_{\Omega_1^0}|\partial_x\big(\mathrm{E}'(w)-\zeta\big)|^2\mathsf{m}(u)\,\mathrm{d}x - c_u[\![\xi]\!]^2\right\}\\
=&\mathcal{M}(0) - \mathcal{M}_*([\![\xi]\!]) = \mathcal{M}_*(0) - \mathcal{M}_*([\![\xi]\!]).
\end{aligned}$$

□

The dual characterization leads to a description of the subdifferential. Let u_ξ solve the minimization problem defining $\mathcal{M}_*([\![\xi]\!])$. Let $c_\xi := \mathrm{harm}\,(\mathsf{m}(u_\xi))$.

Lemma 5.13. *We set* $\kappa_\xi := c_\xi[\![\xi]\!]$ *then* $\dot\mu_{\kappa_\xi} \in \partial\mathcal{R}^{\zeta*}_{\mathrm{memb}}(u_\pm, \xi)$ *with* $\dot\mu_\kappa$ *given in* (5.11).

Proof. We prove the claim by showing

$$\mathcal{R}^{\zeta}_{\mathrm{memb}}(u_\pm, \dot\mu_{\kappa_\xi}) \le \langle\dot\mu_{\kappa_\xi}, \xi\rangle - \mathcal{R}^{\zeta*}_{\mathrm{memb}}(u_\pm, \xi). \tag{5.15}$$

For all $[\![\varphi]\!] \in \mathbb{R}$ we have

$$\begin{aligned}
&\mathcal{M}_*([\![\varphi]\!]) \le \frac{1}{2}c_\xi([\![\xi-\varphi]\!])^2 + \frac{1}{2}\int|\partial_x\mathrm{E}'(u_\xi)|^2\mathsf{m}(u_\xi)\,\mathrm{d}x - \frac{1}{2}c_\xi[\![\varphi]\!]^2\\
\Leftrightarrow&\mathcal{M}_*([\![\varphi]\!]) + c_\xi[\![\xi]\!][\![\varphi]\!] \le \frac{1}{2}c_\xi[\![\xi]\!]^2 + \frac{1}{2}\int|\partial_x\mathrm{E}'(u_\xi)|^2\mathsf{m}(u_\xi)\,\mathrm{d}x\\
\Leftrightarrow&c_\xi[\![\xi]\!][\![\varphi]\!] + \mathcal{M}_*([\![\varphi]\!]) \le c_\xi[\![\xi]\!]^2 + \mathcal{M}_*([\![\xi]\!])
\end{aligned}$$

□

Although we know by Lemma 5.10 that $\mathcal{M}_*(-[\![\mathrm{E}'(u)]\!]) = 0$ since

$$\mathcal{R}^*_{\mathrm{memb}}(u_\pm, -[\![\mathrm{E}'_m(u)]\!]) = \mathcal{M}(0) = \mathcal{M}_*(0),$$

we show directly that $\mathcal{M}_*(-[\![\mathrm{E}'(u)]\!]) = 0$ simply by applying Jensen's estimate with $\mathrm{d}\mathbb{P} = \mathrm{harm}\,(\mathsf{m}(u))\,\frac{1}{\mathsf{m}(u)}\,\mathrm{d}x$ to the integrand $f = \partial_x\big(\mathrm{E}'(u)-\zeta\big)\mathsf{m}(u)$. Hence, we obtain

$$[\![\mathrm{E}'(u)-\zeta]\!]^2 \le \mathrm{harm}\,(\mathsf{m}(u))\int_{\Omega_1^0}\left|\partial_x\big(\mathrm{E}'(u)-\zeta\big)\right|^2\mathsf{m}(u)\,\mathrm{d}x$$

with equality if and only if f is spatially constant, which is exactly the contact condition of Lemma 5.10.

Note that the solution to $\partial_x f = 0$ can be explicitly calculated for $\zeta \equiv 0$. Let F be a primitive of $\mathrm{E}''\mathsf{m}$, then $f = \partial_x\mathrm{F}(u)$ and F is invertible. The solution is given by $u_0(x) = \mathrm{F}^{-1}([\![\mathrm{F}(u)]\!]x + b)$ with $b = [\![\mathrm{F}(u)]\!](1 + x_+ - x_-)/2$ is such that u_0 satisfies the boundary conditions.

In the sequel we are concerned with the minimization problem given by (5.14).

Lemma 5.14. *Let* $F(u) = \int_0^u E''(v)\sqrt{\mathsf{m}(v)}\,\mathrm{d}v$ *satisfy the growth condition* $\mathsf{m} \circ F^{-1}(v) \le c(1 + |v|^{2-\delta})$ *for some* $0 < \delta < 2$ *and let* $\zeta \in \mathrm{C}^1(\Omega_1^0)$*. Then, there exists a global minimizer* $u^0(u^\pm, r; \zeta) \in \mathrm{C}^0(\overline{\Omega}^0)$ *of*

$$\mathcal{I}(u,r) = \frac{1}{2}\int_{\Omega_1^0} \big|\partial_x\big(\mathrm{E}'(u) - \zeta\big)\big|^2 \mathsf{m}(u)\,\mathrm{d}x - \frac{1}{2}\,\mathrm{harm}\,(\mathsf{m}(u))\,r^2$$

subject to the constraint $u(x_\pm) = u^\pm$*.*

Proof. We rewrite the functional in terms of $v = F(u)$ and obtain

$$\begin{aligned}\hat{\mathcal{I}}(v) = \mathcal{I}\big(F^{-1}(v)\big) =& \frac{1}{2}\int_{\Omega_1^0} |\partial_x v|^2\,\mathrm{d}x - \frac{1}{2}\,\mathrm{harm}\,\big(\mathsf{m} \circ F^{-1}(v)\big)\,r^2 \\ &+ \frac{1}{2}\int_{\Omega_1^0} |\partial_x\zeta|^2 \mathsf{m} \circ F^{-1}(v) - 2\sqrt{\mathsf{m} \circ F^{-1}(v)}\,\partial_x v\;\partial_x\zeta\,\mathrm{d}x.\end{aligned}$$

We easily see that $\hat{\mathcal{F}}$ is weakly lower semicontinuous on $\mathrm{H}^1(\Omega_1^0)$ via the compact embedding $\mathrm{H}^1(\Omega_1^0) \Subset \mathrm{C}^0(\overline{\Omega}^0)$.

It remains to show that $\hat{\mathcal{F}}$ is coercive. We use the estimates

$$\hat{\mathcal{I}}(v) \ge \frac{1}{4}\int_{\Omega_1^0} |\partial_x v|^2\,\mathrm{d}x - \frac{1}{2}\int_{\Omega_1^0} |\partial_x\zeta|^2 \mathsf{m} \circ F^{-1}(v)\,\mathrm{d}x - \frac{r^2}{2}\mathsf{m} \circ F^{-1}(v)$$

and $\mathrm{harm}_{\Omega_1^0}(u) \le \int_{\Omega_1^0} u\,\mathrm{d}x$ to obtain

$$\hat{\mathcal{I}}(v) \ge \frac{1}{2}\int_{\Omega_1^0} |v'|^2 - \frac{\|\partial_x\zeta\|_\infty^2 + r^2}{2}\mathsf{m} \circ F^{-1}(v)\,\mathrm{d}x.$$

The growth condition on F^{-1} and the boundary conditions give

$$\hat{\mathcal{I}}(v) \ge \frac{1}{4}\int_{\Omega_1^0} |v'|^2\,\mathrm{d}x - C$$

where $C > 0$ depends on $u^\pm$ via the Poincaré-Friedrich constant satisfying

$$\|v\|_{\mathrm{L}^{2-\delta}}^2 \le c_{\mathrm{PF}}\|v'\|_{\mathrm{L}^{2-\delta}}^2$$

and on r and $\|\partial_x\zeta\|_\infty$.

hence, we have a coercive and weakly lower semi-continuous function on $\mathrm{H}^1(\Omega_1^0)$ and the direct method of calculus of variations gives the existence of a minimizer. □

Example 5.15. *For the special cases* $\mathsf{m}(u) = u^\gamma$ *and* $\mathrm{E} = \mathrm{E}_m$ *we get* $F(u) = \frac{2}{2m+\gamma-2}u^{m-1+\gamma/2}$ *and hence,* $\mathsf{m} \circ F^{-1}(v) = c\;v^{\frac{2\gamma}{2m+\gamma-2}}$*. For* $(m,\gamma) \in \Theta$ *we have* $\gamma \le \frac{2\gamma}{2m+\gamma-2} \le 2\frac{\gamma}{m} < 2$*.*

For $(m,\gamma)\in\Theta$ and $\zeta=0$ we easily see that $u\mapsto\mathcal{I}(u,r)$ is convex for all $r\in\mathbb{R}$.

Lemma 5.16. *The function* $\mathrm{L}^1_{\geq 0}(\Omega^0_1)\ni u\mapsto -\operatorname{harm}_{\Omega^0_1}(\mathsf{m}(u))\in\mathbb{R}$ *is convex. For* $u^{\pm}\geq\delta>0$ *we have strict convexity on* $\{u\in \mathrm{C}^0(\overline{\Omega}^0)|u(x_\pm)=u^\pm\}$

Proof. Let $\delta>0$ and $u,v>\delta$ and thus $\operatorname{harm}_{\Omega^0_1}(\mathsf{m}(u)),\operatorname{harm}_{\Omega^0_1}(\mathsf{m}(v))>\mathsf{m}(\delta)$. Then a supporting hyperplane is given by

$$-\operatorname{harm}_{\Omega^0_1}(\mathsf{m}(v))\geq-\operatorname{harm}_{\Omega^0_1}(\mathsf{m}(u))^2\int_{\Omega^0_1}\frac{\mathsf{m}'(u)(v-u)+\mathsf{m}(u)}{\mathsf{m}(u)^2}\,\mathrm{d}z.$$

This follows from Jensen's estimate for $\mathrm{d}\mathbb{P}=\operatorname{harm}_{\Omega^0_1}(\mathsf{m}(v))\frac{\mathrm{d}z}{\mathsf{m}(v)}$ which reads

$$\operatorname{harm}_{\Omega^0_1}(\mathsf{m}(v))\int_{\Omega^0_1}\left(\frac{\mathsf{m}(v)}{\mathsf{m}(u)}\right)^2\frac{\mathrm{d}z}{\mathsf{m}(v)}\geq\left(\operatorname{harm}_{\Omega^0_1}(\mathsf{m}(v))\int_{\Omega^0_1}\frac{\mathsf{m}(v)}{\mathsf{m}(u)}\frac{\mathrm{d}z}{\mathsf{m}(v)}\right)^2$$

and from $\mathsf{m}(v)\leq\mathsf{m}'(u)(v-u)+\mathsf{m}(u)$. hence, convexity is proved on $\mathrm{L}^1_{\geq\delta}(\Omega^0_1)$ for all $\delta>0$ and thus on $\mathrm{L}^1_{\geq 0}(\Omega^0_1)$. Strict convexity on $\{u\in\mathrm{C}^0(\overline{\Omega}^0)|u(x_\pm)=u^\pm\geq\delta\}$ follows from the fact that

$$-\operatorname{harm}_{\Omega^0_1}(\mathsf{m}(v))=-\operatorname{harm}_{\Omega^0_1}(\mathsf{m}(u))^2\int_{\Omega^0_1}\frac{\mathsf{m}(v)}{\mathsf{m}(u)^2}\,\mathrm{d}z\quad\Leftrightarrow u=v\ \mathbb{P}\text{-a.e.}$$

Here we used $\frac{\mathsf{m}(v)}{\mathsf{m}(u)}\equiv\mathrm{const}$ if and only if $u=v$ due to the boundary conditions. □

Let $u^\pm\geq\delta>0$ and $(\partial_x\zeta)_{|\Omega^0_1}\equiv 0$, then for any $u\in\mathrm{C}^0(\overline{\Omega}^0)$ we have $\mathcal{I}(u,r)\geq\mathcal{I}(\min\{u,\delta\},r)$. Hence, for $u^\pm\geq\delta>0$ we obtain a unique minimizer $u^0\in\mathrm{H}^1(\Omega^0_1)$ such that $\mathcal{M}_*(r)=\mathcal{I}(u^0,r)$

The proof of the recovery sequence involves some approximation arguments. This motivates the study of the dependence of u^0 on r.

Lemma 5.17. *Let* $\delta>0$, $(\partial_x\zeta)_{|\Omega^0_1}\equiv 0$ *and* $u_\pm\geq\delta$. *Let* $r_n\to r$, *then* $u^0(r_n)\rightharpoonup u^0(r)$ *in* $\mathrm{H}^1(I^0)$ *and* $\mathcal{M}_*(u_\pm,r_n)\to\mathcal{M}_*(u_\pm,r)$.

Proof. Note that $\mathcal{M}_*(u_\pm,r_n)\geq\mathcal{M}_*(u_\pm,\sup_n|r_n|)$ and that $\mathcal{I}(\cdot,r_n)\xrightharpoonup{\Gamma}\mathcal{I}(\cdot,r)$ in $\mathrm{H}^1(I^0)$. By standard tools of Γ-convergence, we obtain the claim. □

5.1.3 The Γ-limsup estimate of $\mathfrak{D}_\varepsilon$

This subsection is concerned with the construction of a recovery sequence for $\mathfrak{D}^\zeta_{\mathrm{eff}}$ at $\mu_0\in\mathrm{L}^1\big(0,T;\mathcal{P}(\overline{\Omega}^0_1)\big)$ with $\mu_0=c_0u_0\mathscr{L}_{|\Omega_1\setminus\Omega^0_1}$. Hence, we drop the dependence on ζ of the quantities $\mathcal{R}_{\mathrm{memb}}$ and $\mathcal{M}$. For the construction of a recovery sequence we use the representation

$$\mathcal{R}_{\mathrm{eff}}(\mu_0,\dot\mu_0)=\langle\varphi,\dot\mu_0\rangle-\mathcal{R}^*_{\mathrm{eff}}(\mu_0,\dot\mu_0)$$

for $\xi_0 \in \partial\mathcal{R}_{\text{eff}}(\mu_0, \dot\mu_0)$, i.e., ξ_0 satisfies the continuity equation

$$\langle \varphi, \dot\mu_0 \rangle = \int_{\Omega_1 \setminus \Omega_1^0} a\partial_x\varphi\partial_x\xi_0 c_0 u_0 \,\mathrm{d}x + c_0 \operatorname{harm}_{\Omega_1^0} \big(a\mathsf{m}(u_{[\![\xi_0]\!]})\big) [\![\xi_0]\!][\![\varphi]\!] \tag{5.16}$$

for all $\varphi \in \mathrm{C}^\infty(\Omega_1 \setminus \Omega_1^0)$ (cf. Lemma 5.13). In Theorem 5.19 below we give the precise statement followed by the rigorous proof. But first, we give the main ideas of the proof.

We assume that the density u_0 satisfies the bound $0 < \alpha \le u_0$ for some $\alpha > 0$. By the lower bound we embed the solution to the continuity equation $\xi_0 \in \partial\mathcal{R}_{\text{eff}}(\mu_0, \dot\mu_0)$ into the linear space $\mathrm{L}^2\big(0, T; \mathrm{H}^1(\Sigma) \times \mathrm{H}^1(\Sigma)\big)$ with $\int_\Sigma \xi_0^+ + \xi_0^- \,\mathrm{d}y = 0$.

For fixed $u \in \mathrm{L}^1(\Omega_1)$ we equip the space

$$\mathrm{W}_\varepsilon(u) = \left\{ \xi : \int_{\Omega_1} \xi \mathbf{m}_\varepsilon \,\mathrm{d}x = 0 \quad \text{and} \quad \int_{\Omega_1} |\partial_x \xi|^2 u \,\mathrm{d}x < \infty \right\} \tag{5.17a}$$

with the norm

$$\|\xi\|^2_{\mathrm{W}_\varepsilon(u)} = \int_{\Omega_1} |\partial_x\xi|^2 u \,\mathrm{d}x.$$

We compare $\mathrm{W}_\varepsilon(u)$ to the space H^1_ε defined via

$$\mathrm{H}^1_\varepsilon = \left\{ \xi : \int_{\Omega_1} \xi \mathbf{m}_\varepsilon \,\mathrm{d}x = 0 \quad \text{and} \quad \int_{\Omega_1} |\partial_x \xi|^2 \,\mathrm{d}x < \infty \right\} \tag{5.17b}$$

equipped with the norm

$$\|\xi\|^2_{\mathrm{H}^1_\varepsilon} = \int_{\Omega_1} |\partial_x\xi|^2 \,\mathrm{d}x.$$

If we assume a lower bound $0 < \alpha \le u$, then we deduce $\alpha\|\xi\|^2_{\mathrm{H}^1_\varepsilon} \le \|\xi\|^2_{\mathrm{W}_\varepsilon(u)}$. By the Poincaré estimate (see Lemma A.2) we additionally have for some $c > 0$ independent of ε that

$$\alpha \int_{\Omega_1} \xi^2 \mathbf{m}_\varepsilon \,\mathrm{d}x \le c\alpha\|\xi\|^2_{\mathrm{H}^1_\varepsilon} \le c\|\xi\|^2_{\mathrm{W}_\varepsilon(u)}.$$

hence, we bound the dissipation.

Lemma 5.18. *If the recovery sequence $\mu_\varepsilon = u_\varepsilon\pi_\varepsilon$ also satisfies $\alpha \le u_\varepsilon$ and*

$$\sup_{\varepsilon>0} \int_0^T \int_{\Omega_1} \dot u_\varepsilon^2 \mathbf{m}_\varepsilon \,\mathrm{d}x \,\mathrm{d}t < \infty,$$

then $\dot\mu_\varepsilon$ is bounded in $\mathrm{L}^2\big(0, T; \mathrm{W}_\varepsilon(u_\varepsilon)^*\big)$, i.e.,

$$\sup_{\varepsilon>0} \int_0^T \mathcal{R}_\varepsilon(\mu_\varepsilon, \dot\mu_\varepsilon) \,\mathrm{d}t = \sup_{\varepsilon>0} \mathfrak{D}^{\text{prim}}_\varepsilon(\mu_\varepsilon; [0, T]) < \infty.$$

Thus, we are able to pass to the limit using linear theory. We use the dual characterization of $\mathcal{R}^*_{\mathrm{memb}}$ given in Lemma 5.12, i.e., $U(u_\pm, [\![\xi]\!]) =: u_{[\![\xi_0]\!]}$ from (5.16) is the minimizer of the minimizing problem

$$\tilde{\mathcal{M}}_*(u_\pm, [\![\xi_0]\!]; \zeta) = \frac{c_0}{2} \inf_w \left\{ \int_{\Omega_1^0} |\partial_x \big(\mathrm{E}'(w) - \zeta\big)|^2 a\mathsf{m}(w) \, \mathrm{d}x - \mathrm{harm}\,(a\mathsf{m}(w))\, [\![\xi_0]\!]^2 \right\} \quad (5.18)$$

subject to the boundary conditions $w(x_\pm) = u_0(x_\pm) =: u_\pm$. We recall that

$$\mathcal{R}^*_{\mathrm{eff}}(\mu, \xi) = \mathcal{R}^*_{\mathrm{bulk}}(\mu, \xi) + \mathcal{R}^*_{\mathrm{memb}}(u_\pm, [\![\xi]\!]) \quad \text{and} \quad \mathcal{R}_{\mathrm{eff}}(\mu, \dot\mu) = \sup_\xi \big\{ \langle \xi, \dot\mu \rangle - \mathcal{R}^*_{\mathrm{eff}}(\mu, \xi) \big\}.$$

Let $\xi_{\dot\mu} \in \mathbb{Y}$ with

$$\mathcal{R}_{\mathrm{eff}}(\mu, \dot\mu) = \langle \dot\mu, \xi_{\dot\mu} \rangle - \mathcal{R}^*_{\mathrm{eff}}(\mu, \xi_{\dot\mu}). \quad (5.19)$$

The optimal profile on the middle layer is given by $U(u^\pm, [\![\xi]\!])$, the minimizer of

$$u \mapsto \tilde{\mathcal{M}}_*(u_\pm, [\![\xi]\!])$$

with respect to the boundary conditions $u(x_\pm) = u_\pm$. Note that the minimizer u_1 of $\mathcal{M}_*$ is given in terms of the minimizer u_2 of $\tilde{\mathcal{M}}_*$ via $u_1 \circ Z_a = u_2$ with

$$Z_a(x) = \mathrm{harm}_{\Omega_1^0}(a) \int_{x_-}^x \frac{1}{a} \, \mathrm{d}\hat{x} + x_-.$$

As a recovery sequence we take a piecewise affine in time approximation of

$$u(t) = \begin{cases} u(t) & \text{on } \Omega_1^\pm \\ U\big(u_\pm(t), [\![\xi_{\dot\mu}(t)]\!]\big) & \text{on } \Omega_1^0 \end{cases}$$

More precisely, we define u^n as a piecewise affine interpolation of the locally averaged values, i.e.,

$$u^{(n)}(t_j) = \begin{cases} \tau_j u & \text{on } \Omega_1 \setminus \Omega_1^0 \\ u^0\big(\tau_j u(z_\pm), [\![\tau_j \xi_{\dot\mu}(t)]\!]\big) & \text{on } \Omega_1^0 \end{cases} \quad (5.20)$$

where $t_j = \frac{j}{n} T$ for $0 \le j \le n$ and $t_j = 0$ for $j < 0$ and $t_j = T$ for $j > n$. The mean τ_j for $j \in \{0, \dots, n\}$ is defined via

$$\tau_j h := \fint_{t_{j-1/2}}^{t_{j+1/2}} h \, \mathrm{d}t.$$

We introduce the mass correction

$$\mathrm{m}_\varepsilon^{(n)}(t) = \frac{1}{\int_{\Omega_1} u^{(n)} \, \mathrm{d}\pi_\varepsilon}$$

and obtain the approximate recovery sequence $\mu_\varepsilon^{(n)} := \mathrm{m}_\varepsilon^{(n)} u^{(n)} \pi_\varepsilon$. Note that

$$\int_{\Omega_1 \setminus \Omega_1^0} u^{(n)} \, \mathrm{d}\pi_\varepsilon \to 1 \quad \text{in } \mathrm{W}^{1,p}(0,T) \quad \text{as } \varepsilon \downarrow 0$$

for any $\infty > p \ge 1$ since $u^n_{|\Omega_1^0}$ lies in $\mathrm{W}^{1,\infty}\big(0,T; \mathrm{L}^\infty(\Omega_1^0)\big)$ for n fixed.

It is essential to let $\varepsilon \downarrow 0$ first and then $n \to \infty$. With Lemma 1.2 it follows that there exists n_ε such that $\mu_\varepsilon^{(n_\varepsilon)}$ is a recovery sequence.

Theorem 5.19. *Let* $\mu_0 \in \mathrm{W}^{1,\infty}\big(0,T;\mathrm{H}^1(\Omega_1 \setminus \Omega_1^0)^*\big)$ *be such that* $\alpha \leq u_0 \leq \alpha^{-1}$ *for some* $0 < \alpha < 1$ *and* $\mathfrak{D}^\zeta_{\text{eff}}(\mu_0) < \infty$*. Then there exists a sequence* n_ε *such that* $\mu_\varepsilon^{(n_\varepsilon)} \to \mu$ *in* $\mathrm{H}^1\big(0,T;\mathrm{H}^1(\Omega_1 \setminus \Omega_1^0)^*\big)$ *for all* $\zeta \in \mathrm{C}^1(\Omega_1)$ *with* $(\partial_x\zeta)_{|\Omega_1^0} \equiv 0$ *and*

$$\mathfrak{D}^\zeta_\varepsilon(\mu_\varepsilon^{(n_\varepsilon)}) \longrightarrow \mathfrak{D}^\zeta_{\text{eff}}(\mu).$$

Proof. In the following we write $\rho_0 = \mathsf{u}_0$. Let $\xi_0 \in \partial\mathcal{R}_{\text{eff}}(\mu_0,\dot\mu_0)$. Then it follows $\xi_0 \in \mathrm{L}^\infty\big(0,T;\mathrm{H}^1(\Omega_1 \setminus \Omega_1^0)\big)$. This is an immediate consequence of $\mu_0 \in \mathrm{W}^{1,\infty}\big(0,T;\mathrm{H}^1(\Omega_1 \setminus \Omega_1^0)^*\big)$ and the continuity equation

$$\langle \varphi, \dot\mu_0\rangle = \int_{\Omega_1\setminus\Omega_1^0} a\partial_x\varphi\partial_x\xi_0\rho_0 \,\mathrm{d}\pi_0 + c_0 \operatorname{harm}_{\Omega_1^0}\big(a\rho_{[\![\xi_0]\!]}\big)\,[\![\xi_0]\!][\![\varphi]\!]$$

since $[\![\xi]\!] \mapsto \operatorname{harm}_{\Omega_1^0}(a\rho_\xi)\,[\![\xi]\!]$ is monotone (see Lemma 5.13). In particular, $u^{(n)}_{|\Omega_1^0}$ is essentially bounded. Moreover, due to strong convergence properties oh $\mu_\varepsilon^{(n)}$ and the restriction $(\partial_x\zeta)_{|\Omega_1^0} \equiv 0$ we assume without loss of generality that $\zeta \equiv 0$.

Step 1: We show that $\int_0^T \mathcal{R}_\varepsilon(\mu_\varepsilon^{(n)},\dot\mu_\varepsilon^{(n)})\,\mathrm{d}t$ is bounded. Note that $\alpha \leq u^{(n)} \leq C$. In particular,

$$\int_{\Omega_1} (\dot u^{(n)})^2 \,\mathrm{d}\pi_\varepsilon \leq \frac{C^2 n^2}{T^2}.$$

The mass correction $\big(\int_{\Omega_1} u^{(n)}\,\mathrm{d}\pi_\varepsilon\big)^{-1}$ is bounded in $\mathrm{W}^{1,\infty}(0,T)$ and converges strongly to 1 in $\mathrm{W}^{1,p}(0,T)$ for any $1 \leq p < \infty$. Hence, we conclude that $\mathcal{R}_\varepsilon(\mu_\varepsilon^{(n)},\dot\mu_\varepsilon^{(n)})$ is bounded in $\mathrm{L}^\infty(0,T)$ as $\varepsilon \downarrow 0$ (cf. Lemma 5.18).

Step 2. Passing to the ε-limit in the continuity equation: The solutions to the continuity equation $\xi_\varepsilon^{(n)} \in \partial\mathcal{R}_\varepsilon(\mu_\varepsilon^{(n)},\dot\mu_\varepsilon^{(n)})$ are bounded in $\mathrm{L}^\infty\big(0,T;\mathrm{H}^1(\Omega_1)\big)$. Hence, there exists a weak limit $\xi^{(n)} \in \mathrm{L}^\infty\big(0,T;\mathrm{H}^1(\Omega_1)\big)$. Since $\dot u_\varepsilon^{(n)} \to \dot u^{(n)}$ strongly in $\mathrm{L}^2\big((0,T)\times\Omega_1\big)$ we obtain

$$\int_0^T \langle \xi_\varepsilon^{(n)}, \dot\mu_\varepsilon^{(n)}\rangle \,\mathrm{d}t \to \int_0^T \langle \xi^{(n)}, \dot\mu^{(n)}\rangle \,\mathrm{d}t.$$

Exploiting that $\dot\mu^{(n)} = 0$ on Ω_1^0 and using the continuity equation (5.7) which reads $\xi_\varepsilon^{(n)} \in \partial\mathcal{R}_\varepsilon(\mu_\varepsilon^{(n)},\dot\mu_\varepsilon^{(n)})$ we conclude that

$$\langle \varphi, \dot\mu_0^{(n)}\rangle = \int_{\Omega_1\setminus\Omega_1^0} a\partial_x\varphi\partial_x\xi^{(n)}\;\rho^{(n)} \,\mathrm{d}\pi_0 + c_0 \operatorname{harm}_{\Omega_1^0}\big(a\rho^{(n)}\big)\,[\![\varphi]\!][\![\xi^{(n)}]\!].$$

Hence, we obtain the estimate

$$\begin{aligned}\liminf_{\varepsilon\downarrow 0}\int_0^T \mathcal{R}^*_\varepsilon(\mu_\varepsilon^{(n)},\xi_\varepsilon^{(n)})\,\mathrm{d}t \geq \int_0^T \Bigg\{ &\int_{\Omega_1\setminus\Omega_1^0} \frac{1}{2}a|\partial_x\xi^{(n)}|^2\;\rho^{(n)}\,\mathrm{d}\pi_0 \\ &+ \frac{c_0}{2}\operatorname{harm}_{\Omega_1^0}\big(a\rho^{(n)}\big)\,[\![\xi^{(n)}]\!]^2 \Bigg\}\,\mathrm{d}t.\end{aligned}$$

Step 3. Passing to the ε-limit in the slope: Using Jensen's estimate with respect to the measure $u^{4-2m-\gamma}\,\mathrm{d}t$, i.e., convexity of the slope term $u \mapsto \mathcal{R}_\varepsilon\big(u\pi_\varepsilon, -\mathrm{D}\mathcal{E}_\varepsilon(u\pi_\varepsilon)\big)$, we observe

$$\tau_j\left(\int_{\Omega_1^\pm} |\partial_x \mathrm{E}'_m(u)|^2 \mathsf{m}(u)\,\mathrm{d}x\right) \geq \int_{\Omega_1^\pm} |\partial_x \mathrm{E}'_m(\tau_j u)|^2 \mathsf{m}(\tau_j u)\,\mathrm{d}x.$$

Since for $t \in (t_j, t_{j+1})$ we have that $u^n(t)$ is a convex combination of $u^n(t_j)$ and $u^n(t_{j+1})$ and the map $u \mapsto |\partial_x u|^2 u^{2m-4+\gamma}$ is convex we obtain

$$\begin{aligned}
&|\partial_x \mathrm{E}'_m\big(u^n(t)\big)|^2 \mathsf{m}\big(u^n(t)\big) \\
&\qquad \leq \frac{(t_{j+1}-t)}{(t_{j+1}-t_j)}|\partial_x \mathrm{E}'_m\big(\tau_{j+1}u\big)|^2 \mathsf{m}\big(\tau_{j+1}u\big) + \frac{(t-t_j)}{(t_{j+1}-t_j)}|\partial_x \mathrm{E}'_m\big(\tau_j u\big)|^2 \mathsf{m}\big(\tau_j u\big).
\end{aligned}$$

Thus

$$\int_0^T \int_{\Omega_1} |\partial_x \mathrm{E}'_m\big(u^n(t)\big)|^2 \mathsf{m}\big(u^n(t)\big)\,\mathrm{d}x\,\mathrm{d}t \leq \int_0^T \int_{\Omega_1} |\partial_x \mathrm{E}'_m\big(u(t)\big)|^2 \mathsf{m}\big(u(t)\big)\,\mathrm{d}x\,\mathrm{d}t.$$

Hence, the integral in the bulk part is bounded. On the membrane part we have $\partial_x U\big(\tau_j u_\pm, [\![\tau_j \xi_0]\!]\big) \in \mathrm{L}^2\big((0,T)\times\Omega_1^0\big)$ since $[\![\xi_0]\!] \in \mathrm{L}^\infty(0,T)$ and $\alpha \leq u^\pm \leq \alpha^{-1}$. Thus the slope is well defined and it depends on ε in terms of the mass correction only. Hence, we obtain

$$\lim_{\varepsilon\downarrow 0}\int_0^T \mathcal{R}^*_\varepsilon\big(\mu_\varepsilon^{(n)}, -\mathrm{D}\mathcal{E}_\varepsilon(\mu_\varepsilon^{(n)})\big)\,\mathrm{d}t = \int_0^T \int_{\Omega_1} \frac{1}{2}|\partial_x \mathrm{E}'_m(u^{(n)})|^2 a\rho^{(n)}\,\mathrm{d}\pi_0\,\mathrm{d}t.$$

Step 4. Passing to the n-limit in the continuity equation: We have strong convergence $\dot{u}^n \to \dot{u}_0$ in $\mathrm{L}^2\big(0,T;\mathrm{H}^1(\Omega_1\setminus\Omega_1^0)^*\,\big)$ by Lemma D.5.

Consequently, we obtain that $\xi^{(n)}$ is bounded in $\mathrm{L}^2\big(0,T;\mathrm{H}^1(\Omega_1\setminus\Omega_1^0)\big)$, since for φ with $\int_{\Omega_1\setminus\Omega_1^0}\varphi\,\mathrm{d}x = 0$ we have that

$$|\!|\!|\varphi|\!|\!|^2 := \int_{\Omega_1\setminus\Omega_1^0} a|\partial_x\varphi|^2\ \rho^{(n)}\,\mathrm{d}x + \mathrm{harm}_{\Omega_1^0}\left(a\rho^{(n)}\right)[\![\varphi]\!]^2$$

is equivalent to the $\mathrm{H}^1(\Omega_1\setminus\Omega_1^0)$-norm. Hence, there exists a weak limit $\xi \in \mathrm{L}^2\big(0,T;\mathrm{H}^1(\Omega_1\setminus\Omega_1^0)\big)$. Since $[\![\xi_0]\!] \in \mathrm{L}^\infty(0,T)$, we apply the dominated convergence theorem and obtain $\rho^{(n)}_{|(0,T)\times\Omega_1^0} \to \mathsf{m}\big(U(u_\pm,[\![\xi_0]\!])\big) =: \rho^0_{[\![\xi_0]\!]}$ in $\mathrm{L}^p(0,T\times\Omega_1^0)$ for any $p\geq 1$. In particular, $\mathrm{harm}_{\Omega_1^0}\left(a\rho^{(n)}\right) \to \mathrm{harm}_{\Omega_1^0}\left(a\rho^0\right)$ in $\mathrm{L}^p(0,T)$ for any $p \geq 1$. Hence, we obtain

$$\langle\varphi,\dot{\mu}_0\rangle = \int_{\Omega_1\setminus\Omega_1^0} a\partial_x\varphi\partial_x\xi\ \rho_0\,\mathrm{d}\pi_0 + c_0\,\mathrm{harm}_{\Omega_1^0}\left(a\rho^0\right)[\![\varphi]\!][\![\xi]\!] \tag{5.21}$$

and

$$\begin{aligned}
&\liminf_{n\to\infty}\int_0^T\left\{\int_{\Omega_1\setminus\Omega_1^0}\frac{1}{2}a|\partial_x\xi^{(n)}|^2\ \rho^{(n)}\,\mathrm{d}x + \frac{1}{2}\mathrm{harm}_{\Omega_1^0}\left(a\rho^{(n)}\right)[\![\xi^{(n)}]\!]^2\right\}\mathrm{d}t \\
&\qquad\qquad \geq \int_0^T\left\{\int_{\Omega_1\setminus\Omega_1^0}\frac{1}{2}a|\partial_x\xi|^2\ \rho_0\,\mathrm{d}x + \frac{1}{2}\mathrm{harm}_{\Omega_1^0}\left(a\rho^0\right)[\![\xi]\!]^2\right\}\mathrm{d}t.
\end{aligned}$$

Note that the solution ξ to the equation (5.21) is unique and that ξ_0 is a solution to it since $\mathrm{harm}_{\Omega_1^0}\left(a\rho^0_{[\![\xi_0]\!]}\right)[\![\xi_0]\!] \in \partial\mathcal{R}_{\mathrm{memb}}(u_\pm, \dot\mu_0)$ (cf. (5.15)).

Step 5. Passing to the n-limit in the slope: By strong convergence we immediately obtain

$$\lim_{n\to\infty}\int_0^T\left\{\int_{\Omega_1\setminus\Omega_1^0}\frac{1}{2}a|\partial_x\mathrm{E}_m'(u^{(n)})|^2\mathsf{m}(u^{(n)})\,\mathrm{d}x+\int_{\Omega_1^0}\frac{1}{2}a|\partial_x\mathrm{E}_m'(u^{(n)})|^2\mathsf{m}(u^{(n)})\,\mathrm{d}x\right\}\mathrm{d}t$$
$$=\int_0^T\left\{\int_{\Omega_1\setminus\Omega_1^0}\frac{1}{2}a|\partial_x\mathrm{E}_m'(u)|^2\mathsf{m}(u)\,\mathrm{d}x+\int_{\Omega_1^0}\frac{1}{2}|\partial_x\mathrm{E}_m'(u^0)|^2a\mathsf{m}(u^0)\,\mathrm{d}x\right\}\mathrm{d}t.$$

Step 6. Conclusion. Using Lemma 5.12 we obtain

$$\int_{\Omega_1^0}\frac{1}{2}|\partial_x\mathrm{E}_m'(u^0)|^2a\mathsf{m}(u^0)\,\mathrm{d}\pi_0-\frac{c_0}{2}\,\mathrm{harm}_{\Omega_1^0}\left(a\mathsf{m}(u^0)\right)[\![\xi_0]\!]^2$$
$$=\mathcal{R}_{\mathrm{memb}}(\mu_0,[\![-\mathrm{D}\mathcal{E}_0(\mu_0)]\!])-\mathcal{R}_{\mathrm{memb}}(\mu_0,[\![\xi_0]\!])$$

From step 1-5 we conclude

$$\limsup_{n\to\infty}\limsup_{\varepsilon\downarrow0}\mathfrak{D}_\varepsilon(\mu_\varepsilon^{(n)},[0,T])$$
$$\le\int_0^T\Big\{\langle\xi_0,\dot\mu_0\rangle-\mathcal{R}_{\mathrm{bulk}}(\mu,\xi_0)-\mathcal{R}_{\mathrm{memb}}(\mu_0,[\![\xi_0]\!])+\mathcal{R}_{\mathrm{bulk}}\big(\mu,-\mathrm{D}\mathcal{E}_0(\mu)\big)$$
$$+\mathcal{R}_{\mathrm{memb}}(\mu_0,[\![-\mathrm{D}\mathcal{E}_0(\mu_0)]\!])\Big\}\,\mathrm{d}t=\mathfrak{D}_{\mathrm{eff}}(\mu_0,[0,T]).$$

With Lemma 1.2 the claim follows. □

For $(m,\gamma)\in\Theta$ we prove the validity of the chain-rule under the assumption that $u\in\mathrm{L}^\infty\big((0,T)\times\Omega_1\setminus\Omega_1^0\big)$.

Lemma 5.20 (Chain-rule). *Let* $(m,\gamma)\in\Theta$ *and* $\mathfrak{D}_{\mathrm{eff}}(\mu)<\infty$. *Then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_0(\mu)=\langle\mathrm{D}\mathcal{E}_0(\mu),\dot\mu\rangle.$$

Proof. Since u is bounded, the continuity equation gives that $\dot u\in\mathrm{L}^2\big(0,T;\mathrm{H}^{-1}(\Omega_1\setminus\Omega_1^0\big)$. As an intermediate step we also assume $u\ge\delta>0$. By $2m+\gamma>1$ we obtain $u\in\mathrm{L}^2\big(0,T;\mathrm{H}^1(\Omega_1\setminus\Omega_1^0)\big)$. By [Bre73, Lemma 3.3] we obtain the chain-rule for $\delta<u<1/\delta$ for some $\delta>0$.

For the case $u\ge\delta>0$ we also obtain that $\partial_x\mathrm{E}_m'(u)\in\mathrm{L}^2\big(0,T;\mathrm{H}^1(\Omega_1\setminus\Omega_1^0)\big)$ and $[\![\mathrm{E}_m'(u)]\!]\in\mathrm{L}^2(0,T;\mathbb{R})$. Thus $\mathrm{D}\mathcal{E}_0(\mu)\in\mathrm{L}^1(0,T;\mathscr{B}^*)$. Using [MRS13, Prop. 2.4] we conclude

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{\mathrm{eff}}(\mu)=\langle\mathrm{D}\mathcal{E}_{\mathrm{eff}}(\mu),\dot\mu\rangle.$$

For the general case $u \geq 0$ we define $u_\delta = u+\delta$ with $\delta > 0$. We check that $\mathfrak{D}_{\text{eff}}(\mu_\delta) \leq \mathfrak{D}_{\text{eff}}(\mu) < \infty$. We easily see that $\kappa_\delta = \kappa$ and $\mathbf{v}_\delta = \mathbf{v}\frac{\mathsf{m}(u)}{\mathsf{m}(u_\delta)}$ thus $|\mathbf{v}_\delta|^2\mathsf{m}(u_\delta) \leq |\mathbf{v}|^2\mathsf{m}(u)$, i.e., $\mathcal{R}_{\text{bulk}}(\mu_{\mathbf{v}_\delta}, \dot{\mu}_{\mathbf{v}_\delta}) \leq \mathcal{R}_{\text{bulk}}(\mu_\delta, \dot{\mu}_{\mathbf{v}})$. Moreover, since $2m+\gamma \leq 4$ we have for the slope terms that $\mathcal{R}^*_{\text{bulk}}\big(\mu_\delta, -\mathrm{D}\mathcal{E}_0(\mu_\delta)\big) \leq \mathcal{R}^*_{\text{bulk}}\big(\mu, -\mathrm{D}\mathcal{E}_0(\mu)\big)$. We recall

$$\begin{aligned}\mathcal{R}_{\text{memb}}(u_\pm, \mu_\kappa) &+ \mathcal{R}^*_{\text{memb}}\big(u_\pm, -\mathrm{D}\mathcal{E}_0(\mu)\big) \\ &= \inf_u \frac{c_0}{2}\operatorname{harm}_{\Omega_1^0}(a)\int_{\Omega_1^0} |\partial_x \mathrm{E}_m'(u)|^2\mathsf{m}(u) + \frac{\kappa^2}{\mathsf{m}(u)}\,\mathrm{d}x = \mathcal{M}(u_\pm, \kappa).\end{aligned}$$

Since $2m+\gamma \leq 4$ we find $\mathcal{M}(u_\pm+\delta, \kappa) \leq \mathcal{M}(u_\pm, \kappa)$. Moreover, $\mathrm{E}_m''(u_\delta) \leq \mathrm{E}_m''(u)$ since $m < 2$. Thus $|\mathbf{v}_\delta\partial_x\mathrm{E}_m'(u_\delta)\mathsf{m}(u_\delta)| \leq |\mathbf{v}\partial_x\mathrm{E}_m'(u)\mathsf{m}(u)|$. By dominated convergence the claim follows. $\square$

5.1.4 Convergence of the gradient flows

We emphasize that $\mathcal{R}_{\text{eff}}$ is a priori well defined only for $\mu = u\pi_0$ such that u has well defined traces $u(x_\pm)$ since $\mathcal{R}_{\text{memb}}$ depends on the traces. However, since the limit of the solutions μ_ε satisfies this property at least for almost all $t \in (0,T)$ we pass to the limit in the EDB and find

$$\mathfrak{D}_{\text{eff}}(\mu, [0,T]) + \mathcal{E}_0\big(\mu(T)\big) = \mathcal{E}_0\big(\mu(0)\big).$$

Moreover, since $\liminf\big\{\mathfrak{D}_\varepsilon(\mu_\varepsilon, [0,t]) + \mathcal{E}_\varepsilon\big(\mu_\varepsilon(t)\big)\big\} = \lim\big\{\mathfrak{D}_\varepsilon(\mu_\varepsilon, [0,t]) + \mathcal{E}_\varepsilon\big(\mu_\varepsilon(t)\big)\big\}$ we conclude convergence of the energies for all $t \in [0,T]$. Thus we have EDP-convergence with tilting for tilts $\zeta \in \mathrm{W}^{1,\infty}$ such that $(\partial_x\zeta)_{|\Omega_1^0} \equiv 0$.

By the characterization of the contact set of $\mathcal{R}_{\text{eff}}$ (see Lemma 5.10 and page 85) we find the limiting equation with $\tilde{m} = m+\gamma-1$ that

$$\dot{u} = \operatorname{div}(a\frac{m}{\tilde{m}}\partial_x u^{\tilde{m}}) \quad \text{on } \Omega_1 \setminus \Omega_1^0$$

with the boundary conditions

$$\big(a\partial_x u^{\tilde{m}}\big)(x_\pm) = \operatorname{harm}_{\Omega_1^0}(a)\,[\![u^{\tilde{m}}]\!] \quad \text{and} \quad \big(a\partial_x u^{\tilde{m}}\big)(\pm 1 + x_\pm) = 0.$$

Note that

$$\frac{m}{\tilde{m}}u^{\tilde{m}} = \int_0^u \mathrm{E}''(v)\mathsf{m}(v)\,\mathrm{d}v.$$

5.2 The Boltzmann-nonlinear setting

The second gradient system inducing the porous medium equation is given by (5.1b). As in [DSZ16] the mathematical entropy satisfies $\gamma\mathrm{E}_1'(u) = \log\big(\mathsf{m}(u)\big)$ with $\mathsf{m}(u) = u^\gamma$ in our case. Since $\mathrm{E} = \gamma\mathrm{E}_1$ is a multiple of the Boltzmann entropy we employ similar techniques as in Section 4.2. However, we need to pass to the limit in both u_ε and $\rho_\varepsilon = \mathsf{m}(u_\varepsilon)$. A strong convergence result yields $\lim \mathsf{m}(u_\varepsilon) = \mathsf{m}(\lim u_\varepsilon)$ on the bulk parts. The limiting gradient system is explicitly computable due to the Boltzmann entropy. This is in contrast to Section 5.1. Clearly, as $\gamma \searrow 1$ we obtain at least formally, the classical Wasserstein gradient system for the linear diffusion equation.

Note that $\Phi_\varepsilon' = \varepsilon^{-1}$ on Ω_ε^0 and $\Phi_\varepsilon' = 1$ on $\Omega_\varepsilon \setminus \Omega_\varepsilon^0$. For $\boldsymbol{\mu} \in \mathcal{P}(\overline{\Omega}_\varepsilon)$ with $\mathrm{d}\boldsymbol{\mu} = \mathbf{u}\,\mathrm{d}\boldsymbol{\pi}_\varepsilon$ we compute the push-forward $(\Phi_\varepsilon)_\#\boldsymbol{\mu} =: \mu$ and obtain $\mathrm{d}\mu = \mathbf{m}_\varepsilon u\,\mathrm{d}\pi_\varepsilon$ where $u = \mathbf{u}\circ\Phi_\varepsilon$ with $\mathbf{m}_\varepsilon$ given in (5.3). Moreover, $\mathrm{d}\pi_\varepsilon = c_\varepsilon \mathbf{m}_\varepsilon\,\mathrm{d}x$ with $c_\varepsilon = 1/(2+\varepsilon)$. The gradient system transforms to

$$
\begin{aligned}
\mathrm{X} &= \mathcal{P}(\overline{\Omega}_1),\\
\mathcal{E}_\varepsilon(\mu) &= \begin{cases} \int_{\Omega_1} \gamma \mathrm{E}_1(u)\,\mathrm{d}\pi_\varepsilon & \text{if } \mu = u\pi_\varepsilon,\\ \infty & \text{else,}\end{cases},\\
\mathcal{R}_\varepsilon^*(\mu,\xi) &= \frac{1}{2}\int_{\Omega_1} \mathrm{a}|\partial_x \xi|^2 u^\gamma\,\mathrm{d}\Pi_\varepsilon
\end{aligned}
$$

where we used that $\mathbf{a}_\varepsilon = \mathbf{m}_\varepsilon \mathrm{a}\circ\Phi_\varepsilon$ and denote $\Pi_\varepsilon = c_\varepsilon \mathscr{L}^1_{|\Omega_1}$. We emphasize that $\pi_\varepsilon = c_\varepsilon \mathbf{m}_\varepsilon \mathscr{L}_{|\Omega_1}$ scales differently on the membrane Ω_1^0 than $\Pi_\varepsilon = c_\varepsilon \mathscr{L}^1_{|\Omega_1}$.

The tilted total dissipation potential reads

$$
\mathfrak{D}_\varepsilon^\zeta(\mu;[0,T]) = \int_0^T \int_{\Omega_1} (|\partial_x \log u^\gamma - \partial_x \zeta|^2 + |v|^2)\mathrm{a}u^\gamma\,\mathrm{d}\Pi_\varepsilon\,\mathrm{d}t
$$

with

$$
\langle \varphi, \dot\mu\rangle = \int_{\Omega_1} a\partial_x\varphi\; v\; u^\gamma\,\mathrm{d}\Pi_\varepsilon \quad \text{for all } \varphi \in \mathrm{C}^1(\Omega_1).
$$

5.2.1 Compactness

In the sequel, we derive a priori bounds and compactness results for curves satisfying the natural bound

$$
\sup_\varepsilon \left\{ \mathfrak{D}_\varepsilon(\mu_\varepsilon;[0,T]) + \sup_{t\in[0,T]} \mathcal{E}_\varepsilon\big(\mu_\varepsilon(t)\big)\right\} < \infty. \tag{5.22}
$$

The first result is concerned with a priori bounds resulting from the dual dissipation $\mathfrak{D}_\varepsilon^{\mathrm{dual}}(\mu_\varepsilon;[0,T]) = \int_0^T \mathcal{R}_\varepsilon^*\big(\mu_\varepsilon, -\mathrm{D}\mathcal{E}_\varepsilon(\mu_\varepsilon)\big)\,\mathrm{d}t$.

Lemma 5.21. *Let $1 < \gamma \le 2$ and $\{\mu_\varepsilon\}_\varepsilon \subset \mathcal{P}(\overline{\Omega}_1)$ such that*

$$
\sup_\varepsilon \big\{ \mathfrak{D}_\varepsilon^{\mathrm{dual}}(\mu_\varepsilon;[0,T]) + \sup_{t\in[0,T]} \mathcal{E}_\varepsilon\big(\mu_\varepsilon(t)\big)\big\} < \infty.
$$

Then

$$
\sup_\varepsilon \int_0^T \int_{\Omega_1} u^\gamma{}_\varepsilon + |\partial_x u_\varepsilon|^\gamma + |\partial_x u_\varepsilon^\gamma|\,\mathrm{d}x\,\mathrm{d}t < \infty.
$$

Moreover, $\rho_{\varepsilon|_{\Omega_1^\pm}} := u^\gamma_{\varepsilon|_{\Omega_1^\pm}}$ *is weakly compact in* $\mathrm{L}^1\big(0,T;\mathrm{W}^{1,1}(\Omega_1^\pm)\big)$.

Proof. The Poincaré-Wirtinger estimate gives

$$\int_0^T \int_{\Omega_1^\pm} u^\gamma \,\mathrm{d}x\,\mathrm{d}t \le \int_0^T \left\{ \int_{\Omega_1^\pm} c_{\mathrm{P}} |\partial_x u^{\gamma/2}|^2 \,\mathrm{d}x + |\Omega_1^\pm|^{-1} \left(\int_{\Omega_1^\pm} u^{\gamma/2} \,\mathrm{d}x \right)^2 \right\} \mathrm{d}t$$
$$\overset{\text{(i)}}{\le} \int_0^T \left\{ \int_{\Omega_1^\pm} c_{\mathrm{P}} |\partial_x u^{\gamma/2}|^2 \,\mathrm{d}x + |\Omega_1^\pm|^{1-\gamma} \left(\int_{\Omega_1^\pm} u \,\mathrm{d}x \right)^\gamma \right\} \mathrm{d}t.$$

In particular, $u^{\gamma/2}$ bounded in $\mathrm{L}^2\big(0,T;\mathrm{H}^1(\Omega_1^\pm)\big)$. In (i) we used that $\gamma \le 2$. Using $\partial_x u^\alpha = c_\alpha u^{\alpha-\gamma/2} \partial_x u^{\gamma/2}$ we obtain for $1 \le p < 2$ and any $\Omega' \subset \Omega_1$

$$\int_0^T \int_{\Omega'} |\partial_x u^\alpha|^p \le c_\alpha^p \left(\int_0^T \int_{\Omega'} |\partial_x u^{\gamma/2}|^2 \right)^{p/2} \left(\int_0^T \int_{\Omega'} |u^{\alpha-\gamma/2}|^{\frac{2p}{2-p}} \right)^{\frac{2-p}{2}}. \tag{5.23}$$

Thus choosing $\Omega' = \Omega_1^\pm$ we obtain that $\partial_x u^\alpha$ is bounded in $\mathrm{L}^p\big((0,T)\times\Omega_1^\pm\big)$ for $\alpha \in \frac{\gamma}{2}[1,\frac{2}{p}]$, since u^γ is bounded. In particular, $\partial_x u$ is equi-integrable in $(0,T)\times\Omega_1^\pm$ since $1 < \gamma \le 2$. We recall that (2.2) reads for $x \in \Omega_1^0$

$$|v|(x) \le \int_{\Omega_1^0} |\partial_x v| \,\mathrm{d}x + \int_{\Omega_1^-} |\partial_x v| + |v| \,\mathrm{d}x.$$

Hence, for $v = u^{\gamma/2}$ we conclude

$$\int_{\Omega_1^0} u^\gamma \,\mathrm{d}x \le 3 \int_{\Omega_1^0} |\partial_x u^{\gamma/2}|^2 \,\mathrm{d}x + 3 \int_{\Omega_1^-} |\partial_x u^{\gamma/2}|^2 + u^\gamma \,\mathrm{d}x.$$

Exploiting (5.23) also on the membrane Ω_1^0 for $\alpha = 1$ and $p = \gamma < 2$ we obtain that $\partial_x u$ is bounded in $\mathrm{L}^\gamma\big((0,T)\times\Omega_1\big)$. The case $\gamma = 2$ is immediate by the bound on the dual dissipation. Choosing $\alpha = \gamma$ and $p = 1$ we obtain the bound on $\partial_x u^\gamma$.

It remains to show the weak compactness of ${u^\gamma}_{\varepsilon}|_{\Omega_1^\pm}$ in $\mathrm{L}^1\big(0,T;\mathrm{W}^{1,1}(\Omega_1^\pm)\big)$. According to (2.4) it suffices to show that ${u^\gamma}_{\varepsilon}|_{\Omega_1^\pm}$ is weakly compact in $\mathrm{L}^1\big((0,T)\times\Omega_1^\pm\big)$. Note that by the bound on $\mathfrak{D}_\varepsilon^{\mathrm{dual}}(\mu_\varepsilon;[0,T])$ we obtain boundedness of ${u^\gamma}_{\varepsilon}|_{\Omega_1^\pm}$ in $\mathrm{L}^1\big(0,T;\mathrm{W}^{1,1}(\Omega_1^\pm)\big)$. The bound on the energies give that $u_\varepsilon|_{\Omega_1^\pm}$ is bounded in $\mathrm{L}^\infty\big(0,T;\mathrm{L}^1(\Omega_1^\pm)\big)$. By the estimate

$$\int_0^T \int_{\Omega_1^\pm} u^{\gamma+1} \,\mathrm{d}x\,\mathrm{d}t \le \|u_\varepsilon\|_{\mathrm{L}^\infty\left(0,T;\mathrm{L}^1(\Omega_1^\pm)\right)} \int_0^T \|u\|_{\mathrm{L}^\infty(\Omega_1^\pm)}^\gamma \,\mathrm{d}t$$

we obtain that $u_{\varepsilon}^{\gamma+1}|_{\Omega_1^\pm}$ is bounded in $\mathrm{L}^1\big((0,T)\times\Omega_1^\pm\big)$. In particular, ${u^\gamma}_\varepsilon$ is uniformly integrable. □

The bound on the primal dissipation $\mathfrak{D}_\varepsilon^{\mathrm{prim}}(\mu_\varepsilon;[0,T]) = \int_0^T \mathcal{R}_\varepsilon(\mu_\varepsilon,\dot\mu_\varepsilon)\,\mathrm{d}t$ leads to a priori estimates for $\dot\mu_\varepsilon$.

Lemma 5.22. *Let* $\{\mu_\varepsilon\}_\varepsilon \subset \mathcal{P}(\overline{\Omega}_1)$ *be such that*

$$\sup_\varepsilon \mathfrak{D}^{\mathrm{prim}}_\varepsilon(\mu_\varepsilon;[0,T]) < \infty.$$

Then we have the BV-estimate

$$\exists\, C>0\ \forall\, 0 \le t_0 < \ldots < t_n = T: \quad \sum_{j=1}^n \mathrm{d}_{\mathcal{W}_1}\big(\mu_\varepsilon(t_{j-1}),\mu_\varepsilon(t_j)\big) < C.$$

Moreover, we have that $\dot\mu_\varepsilon$ *is bounded in* $\mathrm{L}^1\big(0,T;\mathrm{W}^{1,\infty}(\Omega_1)^*\big)$*.*

Proof. For any $\xi \in \mathrm{L}^\infty\big(0,T;\mathrm{W}^{1,\infty}(\Omega_1)\big)$ we have

$$\int_0^T \langle \xi, \dot\mu_\varepsilon\rangle \,\mathrm{d}t \le \int_0^T \mathcal{R}_\varepsilon(\mu_\varepsilon,\dot\mu_\varepsilon) + \mathcal{R}^*_\varepsilon(\mu_\varepsilon,\xi)\,\mathrm{d}t.$$

We estimate

$$\int_0^T \mathcal{R}^*_\varepsilon(\mu_\varepsilon,\xi)\,\mathrm{d}t \le \bar{\mathrm{a}} \int_0^T \int_{\Omega_1} u_\varepsilon^\gamma \,\mathrm{d}x\,\mathrm{d}t \|\partial_x \xi\|^2_{\mathrm{L}^\infty\left(0,T;\mathrm{L}^\infty(\Omega_1)\right)}.$$

By Lemma 5.21 we have that

$$\int_0^T \int_{\Omega_1} u_\varepsilon^\gamma \,\mathrm{d}x\,\mathrm{d}t < C$$

for some $C>0$. We use the characterization of the 1-Wasserstein distance via Lipschitz functions (see [AGS05])

$$\mathrm{d}_{\mathcal{W}_1}(\mu_1,\mu_2) = \sup\Big\{ \int_{\Omega_1} \varphi(x)\,\mathrm{d}\mu_1(x) - \int_{\Omega_1} \varphi(x)\,\mathrm{d}\mu_2(x) \,\Big|\, \varphi \in \mathrm{C}^{\mathrm{Lip}}(\Omega_1),\ \mathrm{Lip}(\varphi) \le 1 \Big\}.$$

Taking the supremum over ξ piecewise constant in time we obtain the BV-estimate. Taking the supremum over all $\xi \in \mathrm{L}^\infty\big(0,T;\mathrm{W}^{1,\infty}(\Omega_1)\big)$ we obtain that $\dot\mu_\varepsilon$ is bounded in $\mathrm{L}^\infty\big(0,T;\mathrm{W}^{1,\infty}(\Omega_1)\big)^*$. Note that due to [Roc71] and the regularity of $\dot\mu_\varepsilon$ we have that

$$\sup_{\|\partial_x\xi\|\le 1} \int_0^T \langle \xi,\dot\mu_\varepsilon\rangle\,\mathrm{d}t = \int_0^T \|\dot\mu_\varepsilon\|_{\mathrm{W}^{1,\infty}(\Omega_1)^*}\,\mathrm{d}t.$$

$\square$

Applying Helly's selection principle [DM09], we extract a pointwise convergent subsequence.

Corollary 5.23. *Let* $\{\mu_\varepsilon\}_\varepsilon \subset \mathcal{P}(\overline{\Omega}_1)$ *satisfy the natural bound* (5.22)*. Then there exists a subsequence such that for all* $t \in [0,T]$ *we have* $\mu_\varepsilon(t) \rightharpoonup^* \mu(t)$ *in* $\mathcal{P}(\overline{\Omega}_1)$*.*

Concerning the convergence of the energies we obtain almost everywhere convergence.

Corollary 5.24. *Let* $\{\mu_\varepsilon\}_\varepsilon \subset \mathcal{P}(\overline{\Omega}_1)$ *satisfy the natural bound* (5.22)*. Then up to a subsequence we have*

$$u_{\varepsilon|_{\Omega_1^\pm}} \to u \quad \textit{in} \quad \mathrm{L}^1\big(0,T;\mathrm{L}^\gamma(\Omega_1^\pm)\big) \quad \textit{and} \quad \varepsilon^\alpha u_{\varepsilon|_{\Omega_1^0}} \to 0 \quad \textit{in} \quad \mathrm{L}^m\big((0,T)\times\Omega_1^0\big)$$

for any $\alpha > 0$*. In particular, we have up to extracting a further subsequence that* $\mathcal{E}_\varepsilon\big(\mu_\varepsilon(t)\big) \to \mathcal{E}_0\big(\mu(t)\big)$ *a.e. in* $(0,T)$ *and that the weak limit of* $\rho_{\varepsilon|_{\Omega_1^\pm}}$ *is given by* $u^\gamma|_{\Omega_1^\pm}$*.*

Proof. By Lemma 5.22 we obtain that $\dot{u}_{\varepsilon|_{\Omega_1^\pm}}$ is bounded in $\mathrm{L}^1\big(0,T;\mathrm{W}_0^{1,\infty}(\Omega_1^\pm)^*\big)$. Lemma 5.21 gives that $u_{\varepsilon|_{\Omega_1^\pm}}$ is bounded in $\mathrm{L}^1\big(0,T;\mathrm{W}^{1,\gamma}(\Omega_1^\pm)\big)$. Applying the Aubin-Lion Lemma [Sim87, Cor. 4] we obtain for

$$X = \mathrm{W}^{1,\gamma}(\Omega_1^\pm) \overset{c}{\hookrightarrow} B = \mathrm{L}^\gamma(\Omega_1^\pm) \hookrightarrow Y = \big(\mathrm{W}_0^{1,\infty}(\Omega_1^\pm)\big)^*$$

that $u_{\varepsilon|_{\Omega_1^\pm}}$ is strongly compact in $\mathrm{L}^1(0,T;B) = \mathrm{L}^1\big(0,T;\mathrm{L}^\gamma(\Omega_1^\pm)\big)$. Similarly, we obtain that $\varepsilon u_{\varepsilon|_{\Omega_1^0}}$ is strongly compact in $\mathrm{L}^1\big(0,T;\mathrm{L}^\gamma(\Omega_1^\pm)\big)$ with $\lim \varepsilon u_{\varepsilon|_{\Omega_1^0}} = 0$.

Hence, up to a subsequence we get almost everywhere convergence of $u_{\varepsilon|_{\Omega_1^\pm}}(t)$ in $\mathrm{L}^\gamma(\Omega_1^\pm)$. Lemma 5.21 gives strong convergence $\varepsilon^\alpha u_{\varepsilon|_{\Omega_1^0}} \to 0$ in $\mathrm{L}^\gamma\big((0,T)\times\Omega_1^0\big)$ which gives a.e. convergence of the energies. □

5.2.2 The Γ-liminf estimate of $\mathfrak{D}_\varepsilon$

In the sequel we proof the Γ-liminf estimate for the tilted total dissipation. In a first step we proof a liminf-estimate for the dual $\mathfrak{D}_\varepsilon^{\zeta,\mathrm{dual}}$ and primal $\mathfrak{D}_\varepsilon^{\mathrm{prim}}$ parts separately. But both estimates depend on the membrane. Solving the minimization problem originating from the membrane

$$\mathcal{M}^\zeta(u_\pm,\kappa) = \inf_u \frac{1}{2}\int_{\Omega_1^0} |\partial_x(\mathrm{E}_1'(u^\gamma)-\zeta)|^2 u^\gamma + \frac{\kappa^2}{u^\gamma}\,\mathrm{d}\Pi_\varepsilon$$

leads to an effective dissipation potential. The main difficulties are the limited a priori bounds we have on $\rho_{\varepsilon|_{\Omega_1^0}} = {u^\gamma}_{\varepsilon|_{\Omega_1^0}}$. Hence, we obtain only a measure on the membrane. However, using the disintegration theorem [AGS05, Thm 5.3.1] and the bounded dissipation we assign boundary values to the measure which link it to the densities on the bulk part. We start with investigating properties of the limiting measure $N_0 = \lim \rho_{\varepsilon|_{\Omega_1^0}}$. By the disintegration theorem we obtain for $N_t \in \mathcal{P}(\overline{\Omega}_1)$ such that $N_0(\,\mathrm{d}x,\,\mathrm{d}t) = N_t(\,\mathrm{d}x)N(\,\mathrm{d}t)$, where $N : \mathcal{B}([0,T]) \ni B \mapsto N_0(B\times\overline{\Omega}_1)$. We shortly write $\mathrm{d}N_0 = \mathrm{d}N_t\,\mathrm{d}N$.

Note that for measures N_0 with a density with respect to the Lebesgue measure, i.e., $\mathrm{d}N_0 = u\,\mathrm{d}(x,t)$ we have that

$$\mathrm{d}N = \left(\int_{\Omega_1} u(x,t)\,\mathrm{d}x\right)\mathrm{d}x\,\mathrm{d}t \quad \text{and} \quad \mathrm{d}N_t = \frac{u(x,t)}{\int_{\Omega_1} u\,\mathrm{d}x}\,\mathrm{d}x.$$

In the following we consider the spatial average on the membrane $N^0 : A \mapsto N_0(A \times \overline{\Omega}_1^0)$ for the limit passage in the membrane part. However, we also use the spatial average on the whole domain to relate the traces $u_\pm$ to the measure restricted to the membrane $N_{0|_{[0,T]\times\overline{\Omega}_1^0}}$.

Proposition 5.25. *Let* $1 < \gamma \le 2$ *and* $\{\mu_\varepsilon\}_\varepsilon \subset \mathcal{P}(\overline{\Omega}_1)$ *such that* $\mu_\varepsilon \rightharpoonup \mu$ *in* $\mathrm{L}^\gamma\big([0,T] \times \Omega_1\big)$ *and* $\rho_\varepsilon \Pi_\varepsilon \rightharpoonup^* N_0$ *such that*

$$\sup_\varepsilon \big\{ \mathfrak{D}_\varepsilon^{\mathrm{dual}}(\mu_\varepsilon, [0,T]) + \sup_{t\in[0.T]} \mathcal{E}_\varepsilon\big(\mu_\varepsilon(t)\big) \big\} < \infty.$$

We denote $N : A \mapsto N_0(A \times \Omega_1)$ *and* $N^0 : A \mapsto N_0(A \times \overline{\Omega}_1^0)$ *with* $A \in \mathcal{B}([0,T])$*. Then we have the disintegrations* $\mathrm{d}N_0 = f_t(x)\,\mathrm{d}x\,\mathrm{d}N$ *and* $\mathrm{d}N_{0|_{\overline{\Omega}_1^0}} = g_t(x)\,\mathrm{d}x\,\mathrm{d}N^0$ *with* $f_t \in \mathrm{W}^{1,1}(\Omega_1)$ N*-a.e. and* $g_t \in \mathrm{W}^{1,1}(\Omega_1^0)$ N^0*-a.e. satisfying the relation* $f_{t|_{\overline{\Omega}_1^0}} = g_t \frac{\mathrm{d}N^0}{\mathrm{d}N}$*. On the bulk parts of the domain we have* $\mathrm{d}N_{0|_{\Omega_1\setminus\overline{\Omega_1}^0}} = \rho\,\mathrm{d}\pi_0\,\mathrm{d}t$*. Moreover,*

$$\liminf \int_0^T \int_{\overline{\Omega}_1^0} a \left| \frac{\partial_x \rho_\varepsilon}{\rho_\varepsilon} \right|^2 \rho_\varepsilon \,\mathrm{d}\Pi_\varepsilon \,\mathrm{d}t \ge \int_0^T \int_{\Omega_1^0} a \left| \frac{\partial_x g_t}{g_t} \right|^2 g_t \,\mathrm{d}z \,\mathrm{d}N^0(t) \tag{5.24}$$

and

$$\liminf \int_0^T \int_{\Omega_1\setminus\Omega_1^0} a |\partial_x \rho_\varepsilon|^2 \frac{1}{\rho_\varepsilon} \,\mathrm{d}x\,\mathrm{d}t \ge \int_0^T \int_{\Omega_1\setminus\Omega_1^0} a |\partial_x \rho|^2 \frac{1}{\rho} \,\mathrm{d}x\,\mathrm{d}t. \tag{5.25}$$

Proof. First, we establish weak*-l.s.c. of the slope-term. For $\rho_\varepsilon \Pi_\varepsilon \rightharpoonup^* N_0$ and $\partial_x \rho_\varepsilon \Pi_\varepsilon \rightharpoonup^* H_0$ we show

$$\liminf \int_0^T \int_{\overline{\Omega}_1^0} \left| \frac{\partial_x \rho_\varepsilon}{\rho_\varepsilon} \right|^2 \rho_\varepsilon \,\mathrm{d}\Pi_\varepsilon \,\mathrm{d}t \ge \int_0^T \int_{\overline{\Omega}_1^0} \left| \frac{\mathrm{d}H_0}{\mathrm{d}N_0} \right|^2 \mathrm{d}N_0(x,t). \tag{5.26}$$

We restrict ourselves to the domain $\overline{\Omega}_1^0$ since we already know that $\mathrm{d}N_{0|_{\Omega_1\setminus\overline{\Omega_1}^0}} = \rho\,\mathrm{d}\pi_0\,\mathrm{d}t$. We denote $\overline{\Omega}_{1T}^0 = [0,T] \times \overline{\Omega}_1^0$ and estimate

$$\int_{\overline{\Omega}_{1T}^0} \frac{|\partial_x \rho_\varepsilon|^2}{\rho_\varepsilon} \,\mathrm{d}x\,\mathrm{d}t = \int_{\overline{\Omega}_{1T}^0} \rho_\varepsilon \left(\left| \frac{\partial_x \rho_\varepsilon}{\rho_\varepsilon} \right|^2 + F^2 - F^2 \right) \mathrm{d}x\,\mathrm{d}t \ge \int_{\overline{\Omega}_{1T}^0} \big(2F \partial_x \rho_\varepsilon - F^2 \rho_\varepsilon \big) \,\mathrm{d}x\,\mathrm{d}t$$

Hence, for all $F \in \mathrm{C}^0(\overline{\Omega}_{1T}^0)$ we have

$$\liminf \int_{\overline{\Omega}_{1T}^0} \frac{|\partial_x \rho_\varepsilon|^2}{\rho_\varepsilon} \,\mathrm{d}\Pi_\varepsilon \,\mathrm{d}t \ge \int_{\overline{\Omega}_{1T}^0} 2F \,\mathrm{d}H_0 - \int_{\overline{\Omega}_{1T}^0} F^2 \,\mathrm{d}N_0.$$

If there exists a measurable $\mathcal{A}$ such that $|H_0|(\mathcal{A}) > 0$ but $N_0(\mathcal{A}) = 0$ then there exists a sequence F_n such that

$$\int_{\overline{\Omega}_1^0} 2F_n \,\mathrm{d}H_0 - \int_{\overline{\Omega}_1^0} F_n^2 \,\mathrm{d}N_0 \to \infty.$$

This contradicts the bound on $\mathfrak{D}_\varepsilon^{\mathrm{dual}}$. Thus we conclude $N_0 \gg H_0$. We approximate $\frac{\mathrm{d}H_0}{\mathrm{d}N_0}$ by a sequence F_n and obtain

$$\int_{\overline{\Omega}_1^0} 2F_n \,\mathrm{d}H_0 - \int_{\overline{\Omega}_1^0} F_n^2 \,\mathrm{d}N_0 = \int_{\overline{\Omega}_1^0} \Big(2F_n \frac{\mathrm{d}H_0}{\mathrm{d}N_0} - F_n^2 \Big) \mathrm{d}N_0 \to \int_{\overline{\Omega}_1^0} \left| \frac{\mathrm{d}H_0}{\mathrm{d}N_0} \right|^2 \mathrm{d}N_0.$$

This proves (5.26). Repeating the argument for the domains $(0,T) \times \Omega_1^\pm$ we find (5.25) by weak convergence of $\rho_{\varepsilon|_{\Omega_1^\pm}}$.

It remains to show the estimate (5.24). The following reasoning holds for both $N_t \in \mathcal{P}(\overline{\Omega}_1)$ and $N_t^0 \in \mathcal{P}(\overline{\Omega}_1^0)$ but we elaborate only on the latter. Moreover, we argue only N^0-a.e. (resp. N-a.e.). Using $H_0 \ll N_0$ we easily see that $\mathrm{d}H_{0|_{\overline{\Omega}_1^0}} = \mathrm{d}N_t^{0\prime} \,\mathrm{d}N^0$ with $\frac{\mathrm{d}N_t^{0\prime}}{\mathrm{d}N_t^0} = \frac{\mathrm{d}H_0}{\mathrm{d}N_0}$. By weak*-convergence $N_t^{0\prime}$ and N_t^0 satisfy the following differential relation: $\forall\ \varphi \in \mathrm{C}^0(\overline{\Omega}_{1T}^0)$ with $\partial_x \varphi \in \mathrm{C}^0(\overline{\Omega}_{1T}^0)$ and $\varphi(t, x_\pm) \equiv 0$

$$-\int_{\overline{\Omega}_{1T}^0} \partial_x \varphi \,\mathrm{d}N_t^0(x) \,\mathrm{d}N^0(t) = \int_{\overline{\Omega}_{1T}^0} \varphi \,\mathrm{d}N_t^{0\prime}(x) \,\mathrm{d}N^0(t)$$

By [AFP00, Thm 3.30] we may represent $N_t^{0\prime}$ as a derivative of a BV-function g_t on Ω_1^0 and conclude $\mathrm{d}N_t^0 = g_t \,\mathrm{d}x$. By the relation $\lambda^1 \gg N_t^0 \gg N_t^{0\prime}$ we conclude even $g_t \in \mathrm{W}^{1,1}(\Omega_1^0)$ and $\mathrm{d}N_t^{0\prime} = \partial_x g_t \,\mathrm{d}x$. This proves (4.27). The $\mathrm{W}^{1,1}(\Omega_1)$-function for N_t is denoted by f_t, i.e., $\mathrm{d}N_t = f_t \,\mathrm{d}x$. Thus (5.24) is shown.

The asserted relation between f_t and g_t can be seen as follows. Denoting $N^\pm : A \mapsto N_0(A \times \Omega_1^\pm)$ and $N_t^\pm$ given by the disintegration theorem we verify

$$\mathrm{d}N_t = \mathbb{1}_{\Omega_1^+} \frac{\mathrm{d}N^+}{\mathrm{d}N} \,\mathrm{d}N_t^+ + \mathbb{1}_{\Omega_1^0} \frac{\mathrm{d}N^0}{\mathrm{d}N} \,\mathrm{d}N_t^0 + \mathbb{1}_{\Omega_1^-} \frac{\mathrm{d}N^-}{\mathrm{d}N} \,\mathrm{d}N_t^-.$$

By integration with $\varphi \in \mathrm{C}^0([0,T] \times \overline{\Omega}_1)$ we have

$$\int_{\overline{\Omega}_{1T}} \varphi \,\mathrm{d}N_0 = \int_0^T \sum_{\iota \in \{-,0,+\}} \int_{\Omega_1^\iota} \varphi \,\mathrm{d}N_t^\iota \,\mathrm{d}N^\iota = \int_0^T \int_{\Omega_1} \varphi \,\mathrm{d}N_t \,\mathrm{d}N.$$

Note that $N^\iota \ll N$ since for all $A \in \mathcal{B}([0,T] \times \overline{\Sigma})$ we have $N^\iota(A) \le N(A)$ for $\iota \in \{-,0,+\}$. □

As a corollary we obtain the liminf estimate also when a tilt $\zeta \in \mathrm{W}^{1,\infty}(\Omega_1)$ is present.

Corollary 5.26. *Let* $\zeta \in \mathrm{W}^{1,\infty}(\Omega_1)$ *and* $\mu_\varepsilon \rightharpoonup \mu$ *in* $\mathrm{L}^\gamma\big([0,T] \times \Omega_1\big)$ *and* $\rho_\varepsilon \Pi_\varepsilon \rightharpoonup^* N_0$ *such that*

$$\sup_\varepsilon \big\{ \mathfrak{D}_\varepsilon^{\mathrm{dual}}(\mu_\varepsilon, [0,T]) + \sup_{t \in [0.T]} \mathcal{E}_\varepsilon\big(\mu_\varepsilon(t)\big) \big\} < \infty.$$

Then

$$\liminf \int_0^T \int_{\overline{\Omega}_1^0} a \left| \frac{\partial_x \rho_\varepsilon}{\rho_\varepsilon} - \partial_x \zeta \right|^2 \rho_\varepsilon \,\mathrm{d}\Pi_\varepsilon \,\mathrm{d}t \ge \int_0^T \int_{\overline{\Omega}_1^0} a \left| \frac{\partial_x g_t}{g_t} - \partial_x \zeta \right|^2 g_t \,\mathrm{d}x \,\mathrm{d}N^0(t)$$

and

$$\liminf \int_0^T \int_{\Omega_1\setminus\Omega_1^0} a|\partial_x(\log\rho_\varepsilon - \zeta)|^2\rho_\varepsilon \,\mathrm{d}x\,\mathrm{d}t \geq \int_0^T \int_{\Omega_1\setminus\Omega_1^0} a|\partial_x(\log\rho - \zeta)|^2\rho \,\mathrm{d}x\,\mathrm{d}t.$$

Proof. We observe $|\partial(\log\rho-\zeta)|^2\rho = |\partial\log\rho|^2\rho - 2\partial\zeta\partial\rho + |\partial\zeta|^2\rho$. Hence, by weak* convergence of $\partial_x\rho_\varepsilon$ and ρ_ε and Proposition 5.25 the result follows. □

We obtain the following boundary conditions for g_t in terms of $\rho_\pm := \rho(x_\pm)$. Moreover, we need below the fact that $c_0\sqrt{\rho_-\rho_+}\,\mathrm{d}t = \sqrt{g_t(z^-)g_t(z^+)}\,\mathrm{d}N^0$ with the normalization constant $c_0 = 1/2 = \lim_{\varepsilon\downarrow 0} c_\varepsilon$.

Corollary 5.27. *Let the assumptions of Proposition 5.25 hold true. Then it holds*

$$c_0\rho_\pm\,\mathrm{d}t = g_t(x_\pm)\,\mathrm{d}N^0 \quad \text{and} \quad \rho_\pm g_t(x_\mp) = \rho_\mp g_t(x_\pm)$$
$$\text{and} \quad c_0\sqrt{\rho_-\rho_+}\,\mathrm{d}t = \sqrt{g_t(x_-)g_t(x_+)}\,\mathrm{d}N^0.$$

Proof. Note that on $\Omega_1\setminus\Omega_1^0$ we have $c_0\rho_\pm\,\mathrm{d}x\,\mathrm{d}t = f_t(x)\,\mathrm{d}x\,\mathrm{d}N$. Moreover, the identity

$$c_0\frac{\rho_\pm}{\rho_\mp}\rho_\mp\,\mathrm{d}t = c_0\rho_\pm\,\mathrm{d}t = \frac{\rho_\pm}{\rho_\mp}g_t(x_\mp)\,\mathrm{d}N^0 = g_t(x_\pm)\,\mathrm{d}N^0$$

gives $\frac{\rho_\pm}{\rho_\mp} = \frac{g_t(x_\pm)}{g_t(x_\mp)}$. In particular, $c_0\sqrt{\rho_-\rho_+}\,\mathrm{d}t = \sqrt{g_t(x_-)g_t(x_+)}\,\mathrm{d}N^0$. Thus it follows that $c_0\rho_\pm\,\mathrm{d}t = f_t(x_\pm)\,\mathrm{d}N$. Proposition 5.25 gives the relation $f_t(x_\pm)\,\mathrm{d}N = g_t(x_\pm)\,\mathrm{d}N^0$. Moreover, with $\mathrm{d}N^\pm := g_t(x_\pm)\,\mathrm{d}N^0$ we conclude the identity

$$\begin{aligned}\sqrt{g_t(x_-)g_t(x_+)}\,\mathrm{d}N^0 &= \sqrt{\frac{\mathrm{d}N^-}{\mathrm{d}(N^++N^-)}\frac{\mathrm{d}N^+}{\mathrm{d}(N^++N^-)}}\,\mathrm{d}(N^++N^-) \\ &= \sqrt{\frac{\rho_-}{\rho_++\rho_-}\frac{\rho_+}{\rho_++\rho_-}}\,c_0(\rho_++\rho_-)\,\mathrm{d}t = c_0\sqrt{\rho_-\rho_+}\,\mathrm{d}t.\end{aligned}$$

The relation $\rho_\pm g_t(x_\mp) = \rho_\mp g_t(x_\pm)$ follows from

$$c_0\rho_\pm g_t(x_\mp)\,\mathrm{d}t = g_t(x_\pm)g_t(x_\mp)\,\mathrm{d}N^0 = c_0\rho_\mp g_t(x_\pm)\,\mathrm{d}t.$$

□

For notational convenience we write $g_t^\pm := g_t(x_\pm)$. Thus we have a Γ-$\liminf$ estimate for the dual part of the dissipation depending on N_0 which is connected to $\rho_\pm$ via the disintegration theorem. For the primal dissipation use the Wasserstein theory of [AGS05] introduced in Section 2.2.

Proposition 5.28. *Let* $1 < \gamma \leq 2$ *and* $\{\mu_\varepsilon\}_\varepsilon \subset \mathcal{P}(\overline{\Omega}_1)$ *be such that* $\mu_\varepsilon \rightharpoonup \mu$ *in* $\mathrm{L}^\gamma([0,T]\times\Omega_1)$ *and* $\rho_\varepsilon\Pi_\varepsilon \rightharpoonup^* N_0$ *such that the natural bound* (5.22) *holds. Let* $N^0 \in \mathcal{P}([0,T])$ *and* $g_t : \Omega_1^0 \to \mathbb{R}$ *be defined from* N_0 *as in Proposition 5.25 and*

$g_t^\pm := g_t(x_\pm)$. *Then* $N_{0|\Omega_1\setminus\overline{\Omega}_1^0}$ *is absolutely continuous, i.e.,* $N_{0|\Omega_1\setminus\overline{\Omega}_1^0} = \rho\,\mathrm{d}\Pi_{0|\Omega_1\setminus\overline{\Omega}_1^0}$ *and*

$$\liminf_\varepsilon \int_0^T \mathcal{R}_\varepsilon(\mu_\varepsilon, \dot\mu_\varepsilon)\,\mathrm{d}t \geq \int_0^T \int_{\Omega_1\setminus\Omega_1^0} a|v|^2\rho\,\mathrm{d}\Pi_0\,\mathrm{d}t + \int_{[0,T]\times\overline{\Omega}_1^0} \frac{|\tilde\kappa|^2}{ag_t}\,\mathrm{d}x\,\mathrm{d}N^0$$

where $(v,\tilde\kappa)$ *satisfy the continuity equation*

$$\int_0^T \int_{\Omega_1\setminus\Omega_1^0} \big(a\partial_x\varphi v - \dot\varphi\big)c_0\rho\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \tilde\kappa[\![\varphi]\!]\,\mathrm{d}N^0(t) = 0$$

for all $\varphi \in \mathrm{C}_c^\infty([0,T]\times\Omega_1)$.

Proof. By Lemma 2.2 we obtain a suitable notion of weak convergence and a limit v such that $v_\varepsilon \rightharpoonup v$ satisfying the liminf-estimate

$$\liminf_\varepsilon \int_0^T \int_{\Omega_1} a|v_\varepsilon|^2\rho_\varepsilon\,\mathrm{d}x\,\mathrm{d}t \geq \int_0^T \int_{\Omega_1\setminus\Omega_1^0} a|v|^2\rho\,\mathrm{d}x\,\mathrm{d}t$$

and

$$\liminf_\varepsilon \int_0^T \int_{\Omega_1^0} a|v_\varepsilon|^2\rho_\varepsilon\,\mathrm{d}x\,\mathrm{d}t \geq \int_{[0,T]\times\overline{\Omega}_1^0} a|v|^2\,\mathrm{d}N_0.$$

We also obtain the limit of the continuity equation for $\varphi \in \mathrm{C}_c^\infty([0,T]\times\Omega_1)$

$$\int_0^T \int_{\Omega_1\setminus\Omega_1^0} \big(a\partial_x\varphi v - \dot\varphi\big)\rho\,\mathrm{d}x\,\mathrm{d}t + \int_{[0,T]\times\overline{\Omega}_1^0} a\partial_x\varphi v\,\mathrm{d}N_0 = 0.$$

In particular, using $\mathrm{d}N_0 = g_t\,\mathrm{d}x\,\mathrm{d}N^0$ and testing with φ such that $\mathrm{supp}(\varphi) \subset \Omega_1^0$ we obtain that $\partial_x(avg_t)_{|\Omega_1^0} = 0$ and thus we define the spatial constant $\tilde\kappa(t) := avg_t$. This concludes the proof. □

In the membrane we obtained a contribution from both, the primal part $\liminf_{\varepsilon\downarrow 0} \mathfrak{D}^{\mathrm{prim}}(\mu_\varepsilon;[0,T])$ and the tilted dual part $\liminf_{\varepsilon\downarrow 0} \mathfrak{D}^{\mathrm{dual}\zeta}(\mu_\varepsilon;[0,T])$, that depends on $\mathrm{d}N_0 = g_t(x)\,\mathrm{d}x\,\mathrm{d}N^0$ and reads

$$\mathfrak{F}(g,\tilde\kappa;\zeta) := \frac{1}{2}\int_0^T \int_{\Omega_1^0} a|\partial_x \log g_t - \partial_x\zeta|^2 g_t + \frac{\tilde\kappa^2}{ag_t}\,\mathrm{d}x\,\mathrm{d}N^0(t).$$

The minimization problem $\min \mathfrak{F}(\cdot,\tilde\kappa;\zeta)$ is solved in [LMPR17, Proposition A.2] and it's value can be explicitly calculated.

Lemma 5.29 ([LMPR17, Proposition A.2]). *Denoting* $\kappa = \frac{\tilde\kappa}{\mathrm{harm}_{\Omega_1^0}\left(a\mathrm{e}^\zeta\right)\sqrt{g_t^+g_t^-\mathrm{e}^{-(\zeta^++\zeta^-)}}}$ *with* $\mathrm{harm}_{\Omega_1^0}\left(a\mathrm{e}^\zeta\right)$ *defined in* (4.4) *we have that*

$$\min_g \mathfrak{F}(\cdot,\tilde\kappa;\zeta) = \mathrm{harm}_{\Omega_1^0}\left(a\mathrm{e}^\zeta\right)\sqrt{g_t^+g_t^-\mathrm{e}^{-(\zeta^++\zeta^-)}}\big(\mathscr{C}(\kappa) + \mathscr{C}^*([\![\log(g_t) - \zeta]\!])\big),$$

where the minimum is taken over all g_t *with traces* $g_t(x_\pm) = g_t^\pm$ *and* $\mathscr{C}^*$ *is given by*

$$\mathscr{C}^*(\xi) = 4\big(\cosh(\xi/2) - 1\big).$$

Corollary 4.16 states that $\sqrt{g_t^+ g_t^-}\,\mathrm{d}N^0 = c_0\sqrt{\rho_-\rho_+}\,\mathrm{d}t$ and $g_t^+/g_t^- = \rho_+/\rho_-$. Hence, we arrive at

$$\liminf \mathfrak{D}_\varepsilon^\zeta(\mu_\varepsilon;[0,T]) \geq \int_0^T \Bigg(\int_{\Omega_1\setminus\Omega_1^0} \big(a|\partial_x(\log\rho-\zeta)|^2 + a|v|^2\big)\rho\,\mathrm{d}\Pi_0 + c_0\mathsf{a}(a,\zeta)\big(\mathscr{C}(\kappa)+\mathscr{C}^*([\![\log\rho-\zeta]\!])\big)\Bigg)\,\mathrm{d}t. \quad (5.27)$$

with (v,κ) satisfying the continuity equation

$$\int_0^T \int_{\Omega_1\setminus\Omega_1^0} \big(a\partial_x\varphi v - \dot\varphi\big)\rho\,\mathrm{d}x + \mathsf{a}(a,\zeta)\kappa[\![\varphi]\!]\sqrt{\rho_-\rho_+}\,\mathrm{d}t = 0 \quad (5.28)$$

for all $\varphi =\in \mathrm{C}_c^\infty\big((0,T)\times(\Omega_1\setminus\Omega_1^0)\big)$ with the effective coefficient

$$\mathsf{a}(a,\zeta) := \operatorname{harm}\left(a\mathrm{e}^\zeta\right)\sqrt{\mathrm{e}^{-(\zeta^++\zeta^-)}}. \quad (5.29)$$

The remaining part of this subsection is devoted to weak differentiability of the limiting curve μ and to define the effective dissipation potential. We introduce the space Y which is defined as the closure of $\mathrm{L}^\infty\big(0,T;\mathrm{C}^1(\Omega_1\setminus\Omega_1^0)\big)$ with respect to the norm

$$\|\Theta\|_Y = \|\partial_x\xi\|_{\mathrm{L}^2_\rho} + \|[\![\xi]\!]\|_{\mathrm{L}^{\mathscr{C}^*}_{\sqrt{\rho_+\rho_-}}},$$

where $\Theta = (\partial_x\xi, [\![\xi]\!])$ and for $0\leq w\in \mathrm{L}^1\big((0,T)\times\Omega_1\setminus\Omega_1^0\big)$ resp. $0\leq w\in\mathrm{L}^1(0,T)$ we have

$$\|\Xi\|^2_{\mathrm{L}^2_w} = \int_0^T\int_{\Omega_1\setminus\Omega_1^0}|\Xi|^2 w\,\mathrm{d}x\,\mathrm{d}t \quad \text{and} \quad \|\xi\|_{\mathrm{L}^{\mathscr{C}^*}_w} = \inf\Big\{k>0 : \int_0^T \mathscr{C}^*(\xi/k)w\,\mathrm{d}t \leq 1\Big\}.$$

For an introduction to Orlicz spaces $\mathrm{L}^{\mathscr{C}^*}_w$ we refer to [RZ91]. We just need the fact that there holds a Hölder estimate

$$\int_0^T \kappa[\![\xi]\!]w\,\mathrm{d}t \leq 2\|\kappa\|_{\mathrm{L}^{\mathscr{C}}_w}\|[\![\xi]\!]\|_{\mathrm{L}^{\mathscr{C}^*}_w}$$

and that $\left(\mathrm{L}^{\mathscr{C}}_w\right)^* = \mathrm{L}^{\mathscr{C}^*}_w$ since $\mathscr{C}(2x)\leq C\mathscr{C}(x)$. By (5.27) we have that $v\in\mathrm{L}^2_\rho$ and $\kappa\in\mathrm{L}^{\mathscr{C}}_{\sqrt{\rho_+\rho_-}}$. In particular, using Hölder's estimate we obtain

$$\int_0^T\int_{\Omega_1\setminus\Omega_1^0}\rho\dot\varphi\,\mathrm{d}x\,\mathrm{d}t \leq C(\|v\|_{\mathrm{L}^2_\rho(\Omega_1\setminus\Omega_1^0)} + \|\kappa\|_{\mathrm{L}^{\mathscr{C}}_{\sqrt{\rho_+\rho_-}}(0,T)})\|\Theta\|_Y,$$

where $\Theta = (\partial_x\varphi, [\![\varphi]\!])$ and C depends only on a. Hence, we have $\dot\mu = \dot\rho\,\mathrm{d}\pi_0\,\mathrm{d}t \in Y^*$. We choose the decomposition $\dot\mu = \dot\mu_v + \dot\mu_{\zeta,\kappa}$ where

$$\langle\dot\mu_v,\varphi\rangle = \int_{\Omega_1\setminus\Omega_1^0} a\partial_x\varphi\rho\,\mathrm{d}\pi_0$$

and

$$\langle \dot\mu_{\zeta,\kappa}, \varphi\rangle = c_0 \mathsf{a}(a,\zeta)\kappa[\![\varphi]\!].$$

We define

$$\mathcal{R}_{\mathrm{bulk}}(\mu,\dot\mu) = \begin{cases} \displaystyle\int_{\Omega_1\setminus\Omega_1^0} a|v|^2 \,\mathrm{d}\mu & \text{if } \dot\mu = \dot\mu_v, \\ \infty & \text{else} \end{cases} \tag{5.30}$$

and

$$\mathcal{R}^{\zeta}_{\mathrm{memb}}(u_\pm,\dot\mu) = \begin{cases} \mathsf{a}(a,\zeta)\mathscr{C}(\kappa)\sqrt{\rho_+\rho_-} & \text{if } \dot\mu = \dot\mu_{\zeta,\kappa}, \\ \infty & \text{else.} \end{cases} \tag{5.31}$$

The effective dissipation potential is defined via the inf-convolution of $\mathcal{R}_{\mathrm{bulk}}$ and $\mathcal{R}^{\zeta}_{\mathrm{memb}}$, i.e.,

$$\mathcal{R}^{\zeta}_{\mathrm{eff}}(\mu,\dot\mu) = \inf\left\{\mathcal{R}_{\mathrm{bulk}}(\mu,\dot\mu_v) + \mathcal{R}^{\zeta}_{\mathrm{memb}}(\mu,\dot\mu_{\zeta,\kappa}) \mid \dot\mu = \dot\mu_v + \dot\mu_{\zeta,\kappa}\right\}$$

It is well known that the dual dissipation potential is then given by $\mathcal{R}^*_{\mathrm{bulk}} + \mathcal{R}^{\zeta\ *}_{\mathrm{memb}}$ (see e.g. [AB86, Roc66]). As a consequence of Proposition 5.25, Proposition 5.28 and Lemma 5.29 we obtain

Theorem 5.30. *Let* $1 < \gamma \le 2$ *and* $\{\mu_\varepsilon\}_\varepsilon \subset \mathcal{P}(\overline{\Omega}_1)$ *be such that* $\mu_\varepsilon \rightharpoonup \mu$ *in* $\mathrm{L}^\gamma\big([0,T]\times\Omega_1\big)$*. Moreover, let* (5.22) *hold and let* $\zeta \in \mathrm{C}^1(\Omega_1 \setminus \overline{\Omega}_1^0)$*. Then* $\mu(\Omega_1^0) = 0$ *and*

$$\liminf_{\varepsilon\downarrow 0} \mathfrak{D}^{\zeta}_{\varepsilon}(\mu_\varepsilon;[0,T]) \ge \mathfrak{D}^{\zeta}_{\mathrm{eff}}(\mu;[0,T])$$

with

$$\mathfrak{D}^{\zeta}_{\mathrm{eff}}(\mu;[0,T]) = \int_0^T \mathcal{R}^{\zeta}_{\mathrm{eff}}(\mu,\dot\mu) + \mathcal{R}^{\zeta*}_{\mathrm{eff}}\big(\mu, -\mathrm{D}\mathcal{E}_0(\mu)\big)\,\mathrm{d}t.$$

5.2.3 The Γ-limsup estimate of $\mathfrak{D}_\varepsilon$

This subsection is concerned with the construction of a recovery sequence for $\mathfrak{D}^{\zeta}_{\mathrm{eff}}$ at $\mu_0 \in \mathrm{L}^1\big(0,T;\mathcal{P}(\overline{\Omega}_1\setminus\Omega_1^0)\big)$ with $\mu_0 = c_0 u_0 \,\mathrm{d}\pi_0$. We denote $\rho_0 = u_0^\gamma$ and $\rho_0^\pm = \rho_0(x_\pm)$. For the construction we use the representation

$$\mathcal{R}^{\zeta}_{\mathrm{eff}}(\mu_0,\dot\mu_0) = \langle\varphi,\dot\mu_0\rangle - \mathcal{R}^{\zeta*}_{\mathrm{eff}}(\mu_0,\dot\mu_0)$$

for $\xi_0 \in \partial\mathcal{R}^{\zeta}_{\mathrm{eff}}(\mu_0,\dot\mu_0)$, i.e., ξ_0 satisfies the continuity equation

$$\langle\varphi,\dot\mu_0\rangle = \int_{\Omega_1\setminus\Omega_1^0} a\partial_x\varphi\partial_x\xi_0\ \rho_0\,\mathrm{d}\pi_0 + c_0\mathsf{a}(a,\zeta)\mathscr{C}^{*\prime}([\![\xi_0]\!])[\![\varphi]\!]\sqrt{\rho_0^+\rho_0^-}\,\mathrm{d}y \tag{5.32}$$

for all $\varphi \in \mathrm{C}^\infty(\Omega_1\setminus\overline{\Omega}_1^0))$ with $\mathsf{a}(a,\zeta)$ given in (5.29). In Theorem 5.32 below we give the precise statement followed by the rigorous proof. But first, we give characterizations for $\mathcal{R}^{\zeta*}_{\mathrm{memb}}$ and main ideas of the rigorous proof.

We assume that the density u_0 satisfies the bound $0 < \alpha \le u_0$ for some $\alpha > 0$. By the lower bound we embed the solution to the continuity equation

$\xi_0 \in \partial\mathcal{R}^{\zeta}_{\text{eff}}(\mu_0, \dot\mu_0)$ into the linear space $\mathrm{L}^2\big(0,T;\mathrm{H}^1(\Omega_1 \setminus \overline{\Omega}^0_1)\big)$ with $\int_{\Omega_1\setminus\overline{\Omega}^0_1} \xi_0 \,\mathrm{d}x = 0$ such that the jump $[\![\xi_0]\!] = \xi_0(x_+) - \xi_0(x_-)$ is in $\mathrm{L}^{\mathscr{C}^*}(0,T)$. For fixed $u \in \mathrm{L}^1(\Omega_1)$ we equip the space

$$\mathrm{W}_\varepsilon(u) = \left\{ \xi : \int_{\Omega_1} \xi \mathbf{m}_\varepsilon \,\mathrm{d}x = 0 \quad \text{and} \quad \int_{\Omega_1} |\partial_x \xi|^2 u \,\mathrm{d}x < \infty \right\} \tag{5.33a}$$

with the norm

$$\|\xi\|^2_{\mathrm{W}_\varepsilon(u)} = \int_{\Omega_1} a|\partial_x\xi|^2 u \,\mathrm{d}x.$$

We compare $\mathrm{W}_\varepsilon(u)$ to the space H^1_ε defined via

$$\mathrm{H}^1_\varepsilon = \left\{ \xi : \int_{\Omega_1} \xi \mathbf{m}_\varepsilon \,\mathrm{d}x = 0 \quad \text{and} \quad \int_{\Omega_1} |\partial_x \xi|^2 \,\mathrm{d}x < \infty \right\} \tag{5.33b}$$

equipped with the norm

$$\|\xi\|^2_{\mathrm{H}^1_\varepsilon} = \int_{\Omega_1} a|\partial_x\xi|^2 \,\mathrm{d}x.$$

If we assume a lower bound $0 < \alpha \le u$, then we deduce $\alpha\|\xi\|^2_{\mathrm{H}^1_\varepsilon} \le \|\xi\|^2_{\mathrm{W}_\varepsilon(u)}$. By the Poincaré estimate (see Lemma A.2) we additionally have for some $c > 0$ independent of ε that

$$\alpha \int_{\Omega_1} \xi^2 \mathbf{m}_\varepsilon \,\mathrm{d}x \le c\alpha \|\xi\|^2_{\mathrm{H}^1_\varepsilon} \le c\|\xi\|^2_{\mathrm{W}_\varepsilon(u)}.$$

hence, we bound the dissipation.

Lemma 5.31. *If the recovery sequence $\mu_\varepsilon = u_\varepsilon \pi_\varepsilon$ also satisfies $\alpha \le u_\varepsilon$ and*

$$\sup_{\varepsilon>0} \int_0^T \int_{\Omega_1} \dot u_\varepsilon^2 \mathbf{m}_\varepsilon \,\mathrm{d}x\,\mathrm{d}t < \infty,$$

then $\dot\mu_\varepsilon$ is bounded in $\mathrm{L}^2\big(0,T;\mathrm{W}_\varepsilon(u_\varepsilon)^*\big)$, i.e.,

$$\sup_{\varepsilon>0} \int_0^T \mathcal{R}_\varepsilon(\mu_\varepsilon, \dot\mu_\varepsilon)\,\mathrm{d}t = \sup_{\varepsilon>0} \mathfrak{D}^{\text{prim}}_\varepsilon(\mu_\varepsilon; [0,T]) < \infty.$$

Thus we are able to pass to the limit using linear theory. In order to find the density also on the membrane Ω^0_1 we look at two different characterization of $\mathcal{R}^{\zeta*}_{\text{memb}}$. Computing the Legendre transform of $\mathcal{R}^{\zeta}_{\text{memb}}$ give in (5.31) we immediately obtain

$$\mathcal{R}^{\zeta*}_{\text{memb}}(\mu_0, [\![\xi]\!]) = c_0 \mathsf{a}(a,\zeta)\mathscr{C}^*([\![\xi]\!])\sqrt{\rho_0^-\rho_0^+}.$$

On the other hand, by Lemma 5.29 we have that $\mathcal{R}^{\zeta}_{\mathrm{memb}}$ is given via a minimization problem, i.e., for $\dot\mu$ such that $\langle\varphi,\dot\mu\rangle = \tilde\kappa[\![\varphi]\!]$ we have

$$\int_0^T \mathcal{R}^{\zeta}_{\mathrm{memb}}(\mu_0,\dot\mu_0)\,\mathrm{d}t = \int_0^T \inf_{g_t}\frac{1}{2}\int_{\overline{\Omega}_1^0} a|\partial_x \log g_t - \partial_x\zeta|^2 g_t + \frac{\tilde\kappa^2}{a g_t}\,\mathrm{d}x\,\mathrm{d}N^0 \\ - \int_0^T \mathcal{R}^{\zeta *}_{\mathrm{memb}}(\mu_0, -[\![\log(g_t)]\!])\,\mathrm{d}t$$

subject to the boundary conditions $c_0\rho_0^{\pm}dt = g_t(x_\pm)\,\mathrm{d}N^0$. Hence, the Legendre transform can be represented by the formula

$$\begin{aligned}\int_0^T \mathcal{R}^{\zeta *}_{\mathrm{jump}}(\mu_0,[\![\xi_0]\!])\,\mathrm{d}t &= \sup_{\tilde\kappa}\Bigg\{\int_0^T \tilde\kappa[\![\xi_0]\!]\,\mathrm{d}N^0 \\ &\qquad - \int_0^T \inf_{g_t}\frac{1}{2}\int_{\overline{\Omega}_1^0} a|\partial_x \log g_t - \partial_x\zeta|^2 g_t + \frac{\tilde\kappa^2}{a g_t}\,\mathrm{d}x\,\mathrm{d}N^0\Bigg\} + C_0 \\ &= C_0 - \frac{1}{2}\inf_{\tilde\kappa, g_t}\Bigg\{\int_0^T \frac{1}{\mathrm{harm}_{\overline{\Omega}_1^0}(a g_t)}\Big(\tilde\kappa - \mathrm{harm}_{\overline{\Omega}_1^0}(a g_t)\,[\![\xi]\!]\Big)^2\,\mathrm{d}N^0 \\ &\qquad + \int_0^T\int_{\overline{\Omega}_1^0} a|\partial_x \log g_t - \partial_x\zeta|^2 g_t - \mathrm{harm}_{\overline{\Omega}_1^0}(a g_t)\,[\![\xi]\!]^2\,\mathrm{d}x\,\mathrm{d}N^0\Bigg\} \\ &= C_0 - \frac{1}{2}\inf_{g}\Bigg\{\int_0^T\int_{\overline{\Omega}_1^0} a|\partial_x \log g_t - \partial_x\zeta|^2 g_t \\ &\qquad - \mathrm{harm}_{\overline{\Omega}_1^0}(a g_t)\,[\![\xi]\!]^2\,\mathrm{d}x\,\mathrm{d}N^0\Bigg\},\end{aligned} \tag{5.34}$$

where we set $C_0 := \int_0^T \mathcal{R}^{\zeta *}_{\mathrm{memb}}(\mu, -[\![\log(g_t)]\!])\,\mathrm{d}t$. In the regular case

$$\mathrm{d}N^0 = \left(\int_{\overline{\Omega}_1^0} c_0\rho_0(x,t)\,\mathrm{d}x\right)\mathrm{d}y\,\mathrm{d}t \quad \text{and} \quad g_t(x) = \frac{\rho_0(x,t)}{\int_{\overline{\Omega}_1^0}\rho_0(x,t)\,\mathrm{d}x},$$

we have to minimize the function

$$\rho \mapsto \int_{\overline{\Omega}_1^0} |\partial_x(\mathrm{E}_1'(\rho)-\zeta)|^2 a\rho\,\mathrm{d}x - \mathrm{harm}_{\overline{\Omega}_1^0}(a\rho)\,[\![\xi_0]\!]^2 \tag{5.35}$$

subject to the boundary condition $\rho(x_\pm) = \rho_0^{\pm}$. The minimizer is denoted by $R(\rho_0^{\pm},[\![\xi_0]\!]) =: \big(U(u_0^{\pm},[\![\xi_0]\!])\big)^{\gamma}$, is explicitly calculated in Lemma B.3 and is the desired density on the membrane, i.e., we define the density on the whole domain Ω_1 via

$$u(x) = \begin{cases} u_0(x) & \text{for } x\in\Omega_1\setminus\Omega_1^0, \\ U(u_0^{\pm},[\![\xi_0]\!])(x) & \text{for } x\in\Omega_1^0. \end{cases}$$

Since the proof of Theorem 5.32 involves several approximations of u we outline the main ideas first.

Combined with the normalization $u_\varepsilon = \frac{u}{\int_{\Omega_1} u \,\mathrm{d}\pi_\varepsilon}$ we obtain that $\mu_\varepsilon = u_\varepsilon \pi_\varepsilon$ satisfies the mass constraint $\mu_\varepsilon(\Omega_1) = 1$. Note that $\int_{\Omega_1} u \,\mathrm{d}\pi_\varepsilon \to 1$ and hence, $\mu_\varepsilon \to \mu$. The continuity equation reads

$$\langle \varphi, \dot\mu_\varepsilon \rangle = \int_{\Omega_1} a \partial_x \varphi \partial_x \xi_\varepsilon \rho_\varepsilon \,\mathrm{d}\pi_\varepsilon.$$

Passing to the limit in this continuity equation we obtain for $\varphi \in \mathrm{C}^1(\Omega_1)$ that $\hat\xi = \lim \xi_\varepsilon$ satisfies

$$\langle \varphi, \dot\mu_0 \rangle = \int_{\Omega_1} a \partial_x \varphi \partial_x \hat\xi \rho_0 \,\mathrm{d}\Pi_0.$$

Testing with φ such that $\operatorname{supp}(\varphi) \subset \Omega_1^0$ we find that $a\partial_x \hat\xi \;\rho = \operatorname{harm}_{\Omega_1^0}(a\rho)\, [\![\hat\xi]\!]$ and consequently

$$\langle \varphi, \dot\mu_0 \rangle = \int_{\Omega_1 \setminus \Omega_1^0} a \partial_x \varphi \partial_x \hat\xi \rho_0 \,\mathrm{d}\pi_0 + c_0 \operatorname{harm}_{\overline{\Omega}_1^0}(a\rho)\, [\![\hat\xi]\!][\![\varphi]\!].$$

Note that $\rho = u^\gamma$ is given on the membrane via $R(\rho_0^\pm, [\![\xi_0]\!])$. By Lemma B.3 we have that

$$\operatorname{harm}_{\Omega_1^0}(a\rho)\, [\![\xi_0]\!] = \mathsf{a}(a,\zeta) \mathscr{C}^{*\prime}([\![\xi_0]\!]) \sqrt{\rho_0^+ \rho_0^-}.$$

In particular, by uniqueness we find that $\hat\xi = \xi_0$ is the solution to (5.32). We recall

$$\mathcal{R}^*_{\mathrm{bulk}}(\mu_0, \xi_0) = \int_{\Omega_1 \setminus \Omega_1^0} \frac{1}{2} a \partial_x \xi_0 \partial_x \xi_0 \rho_0 \,\mathrm{d}\pi_0.$$

Hence

$$\begin{aligned}\limsup_{\varepsilon\downarrow 0} \mathcal{R}_\varepsilon(\mu_\varepsilon, \dot\mu_\varepsilon) &= \limsup_{\varepsilon\downarrow 0}\, \langle \xi_\varepsilon, \dot\mu_\varepsilon \rangle - \mathcal{R}^*_\varepsilon(\mu_\varepsilon, \xi_\varepsilon) = \langle \xi, \dot\mu \rangle - \liminf_{\varepsilon\downarrow 0} \mathcal{R}^*_\varepsilon(\mu_\varepsilon, \xi_\varepsilon) \\ &\le \langle \xi_0, \dot\mu \rangle - \mathcal{R}^*_{\mathrm{diff}}(\mu_0, \xi_0) - \frac{1}{2} \operatorname{harm}_{\Omega_1^0}(a\rho)\, [\![\xi_0]\!]^2.\end{aligned}$$

Together with

$$\lim_{\varepsilon\downarrow 0} \mathcal{R}_\varepsilon\big(\mu_\varepsilon, -\mathrm{D}\mathcal{E}_\varepsilon(\mu_\varepsilon)\big) = \mathcal{R}^*_{\mathrm{diff}}\big(\mu_0, -\mathrm{D}\mathcal{E}_0(\mu_0)\big) + \int_{\Omega_1^0} \frac{1}{2} |\partial_x \mathrm{E}_1'(u)|^2 a u \,\mathrm{d}x$$

we obtain that

$$\begin{aligned}&\limsup_{\varepsilon\downarrow 0} \Big(\mathcal{R}_\varepsilon(\mu_\varepsilon, \dot\mu_\varepsilon) + \mathcal{R}_\varepsilon\big(\mu_\varepsilon, -\mathrm{D}\mathcal{E}_\varepsilon(\mu_\varepsilon) + \zeta\big)\Big) \\ &\qquad\qquad \le \langle \xi_0, \dot\mu \rangle - \mathcal{R}^{\zeta *}_{\mathrm{eff}}(\mu, \xi_0) + \mathcal{R}^{\zeta *}_{\mathrm{eff}}\big(\mu, -\mathrm{D}\mathcal{E}_0(\mu) + \zeta\big).\end{aligned}$$

Here we used (5.34), i.e.,

$$\begin{aligned}&\int_{\Omega_1^0} \frac{1}{2} |\partial_x \mathrm{E}_1'(\rho) - \partial_x \zeta|^2 a\rho \,\mathrm{d}x - \frac{1}{2} \operatorname{harm}_{\overline{\Omega}_1^0}(a\rho)\, [\![\xi_0]\!]^2 \\ &\qquad\qquad = \mathcal{R}^{\zeta *}_{\mathrm{jump}}\big(\mu, [\![-\mathrm{D}\mathcal{E}_0(\mu) + \zeta]\!]\big) - \mathcal{R}^{\zeta *}_{\mathrm{jump}}\big(\mu, [\![\xi_0]\!]\big).\end{aligned}$$

Note that $\xi_0 \in \partial\mathcal{R}_{\text{eff}}(\mu,\dot\mu)$ and thus

$$\limsup_{\varepsilon\downarrow 0}\Big(\mathcal{R}_\varepsilon(\mu_\varepsilon,\dot\mu_\varepsilon)+\mathcal{R}_\varepsilon\big(\mu_\varepsilon,-\mathrm{D}\mathcal{E}_\varepsilon(\mu_\varepsilon)+\zeta\big)\Big)\le \mathcal{R}_{\text{eff}}(\mu,\dot\mu)+\mathcal{R}^*_{\text{eff}}\big(\mu,-\mathrm{D}\mathcal{E}_0(\mu)+\zeta\big).$$

In order to justify the steps above we need to modify the construction in two ways. Note that the solution $R(\rho_0^\pm,[\![\xi_0]\!])$ to the minimization problem given in (B.2) is not necessarily weakly differentiable in time hence, $\dot\mu_\varepsilon$ is not well defined. Moreover, the bound on the effective dissipation gives $R(\rho_0^\pm,[\![\xi_0]\!])\in \mathrm{L}^1\big((0,T)\times\Omega_1^0\big)$ only. To gain integrability we need to truncate $[\![\xi_0]\!]$.

In order to bound $\mathcal{R}_\varepsilon(\mu_\varepsilon,\dot\mu_\varepsilon)$ we discretize. Let $t_j=\frac{j}{n}T$ for $0\le j\le n$ and $t_j=0$ for $j<0$ and $t_j=T$ for $j>n$. We define the mean τ_j for $j\in\{0,\dots,n\}$ and the truncation χ_m for $m>0$ via

$$\tau_j h := \fint_{t_{j-1/2}}^{t_{j+1/2}} h\,\mathrm{d}t \quad\text{ and }\quad \chi_m(\ell)=\max\big\{\min\{m,\ell\},-m\big\}.$$

We define $u^{(n,k)}$ at the nodal points $\{t_j\}$ via

$$u^{(n,k)}(t_j)=\begin{cases}\tau_j u_0^\pm & \text{on } \Omega_1^\pm,\\ u^0\big(\tau_j u_0^\pm,\chi_k([\![\tau_j\xi_0]\!])\big) & \text{on } \Omega_1^0\end{cases}$$

and via affine interpolation in between. The density then reads $u_\varepsilon^{(n,k)}=\frac{u^{(n,k)}}{\int_{\Omega_1}u^{(n,k)}\,\mathrm{d}\pi_\varepsilon}$.

Theorem 5.32. *Let $1>\alpha>0$ and μ_0 such that $\mathfrak{D}_{\text{eff}}(\mu_0,[0,T])<\infty$ with density u_0 satisfying $\alpha\le u_0\le\alpha^{-1}$ and $\dot\mu_0\in\mathrm{L}^2\big(0,T;\big(\mathrm{H}^1(\Sigma)\times\mathrm{H}^1(\Sigma)\big)^*\big)$. Then there exist sequences $n_\varepsilon,m_\varepsilon\to\infty$ as $\varepsilon\downarrow 0$ such that*

$$\limsup_{\varepsilon\downarrow 0}\mathfrak{D}_\varepsilon^\zeta(\mu_\varepsilon^{n_\varepsilon,m_\varepsilon},[0,T])\le\mathfrak{D}^\zeta_{\text{eff}}(\mu_0,[0,T]).$$

Proof. For the limit passage in the slope term (step 3,5 and 7) we use that $\mathcal{R}^*_\varepsilon\big(\mu_\varepsilon^{(n,k)},-\mathrm{D}\mathcal{E}_\varepsilon(\mu_\varepsilon^{(n,k)})+\zeta\big)-\mathcal{R}^*_\varepsilon\big(\mu_\varepsilon^{(n,k)},-\mathrm{D}\mathcal{E}_\varepsilon(\mu_\varepsilon^{(n,k)})\big)$ depends linearly on $u_\varepsilon^{(n,k)}$. Hence, it suffices to pass to the limit in $\mathcal{R}^*_\varepsilon\big(\mu_\varepsilon^{(n,k)},-\mathrm{D}\mathcal{E}_\varepsilon(\mu_\varepsilon^{(n,k)})\big)$ only.

Step 1: We show that $\int_0^T\mathcal{R}_\varepsilon(\mu_\varepsilon^{(n,k)},\dot\mu_\varepsilon^{(n,k)})\,\mathrm{d}t$ is bounded. Note that $\alpha\le u^{(n,k)}\le 2\alpha^{-1}\big(1+\cosh(k/2)\big)\mathrm{e}^{\|\zeta\|_\infty}$. In particular,

$$\int_{\Omega_1}(\dot u^{(n,k)})^2\,\mathrm{d}\pi_\varepsilon\le\frac{4\alpha^{-2}\big(1+\cosh(k/2)\big)^2n^2\mathrm{e}^{2\|\zeta\|_\infty}}{T^2}.$$

The mass correction $\big(\int_{\Omega_1}u^{(n,k)}\,\mathrm{d}\pi_\varepsilon\big)^{-1}$ is bounded in $\mathrm{W}^{1,\infty}(0,T)$ and converges strongly to 1 in $\mathrm{W}^{1,p}(0,T)$ for any $1\le p<\infty$. Hence, by Lemma 5.31 we have that $\mathcal{R}_\varepsilon(\mu_\varepsilon^{(n,k)},\dot\mu_\varepsilon^{(n,k)})$ is bounded in $\mathrm{L}^1(0,T)$ as $\varepsilon\downarrow 0$.

Step 2. Passing to the ε-limit in the continuity equation: Note that as $\varepsilon\downarrow 0$ the limit measure $\mu_0^{(n)}$ does only depend on n. Only the limit of the densities on the membrane depend on k. There exists a limit of the solutions to the continuity equation $\xi_\varepsilon^{(n,k)} \in \partial\mathcal{R}_\varepsilon(\mu_\varepsilon^{(n,k)}, \dot\mu_\varepsilon^{(n,k)})$ since $\dot\mu_\varepsilon^{(n,k)}$ is bounded in $\mathrm{L}^2\big(0,T;\mathrm{H}^1(\Omega_1)^*\big)$. Since $\dot u_\varepsilon^{(n,k)} \to \dot u^{(n)}$ strongly in $\mathrm{L}^2\big((0,T)\times\Omega_1\big)$ we obtain

$$\int_0^T \langle \xi_\varepsilon^{(n,k)}, \dot\mu_\varepsilon^{(n,k)}\rangle \,\mathrm{d}t \to \int_0^T \langle \xi^{(n,k)}, \dot\mu^{(n)}\rangle \,\mathrm{d}t.$$

Exploiting that $\dot\mu^{(n,k)} = 0$ on Ω_1^0 and passing to the limit in the continuity equation $\xi_\varepsilon^{(n,k)} \in \partial\mathcal{R}_\varepsilon(\mu_\varepsilon^{(n,k)}, \dot\mu_\varepsilon^{(n,k)})$ we conclude that

$$\langle \varphi, \dot\mu_0^{(n)}\rangle = \int_{\Omega_1\setminus\Omega_1^0} a\partial_x\varphi\partial_x\xi^{(n,k)}\; c_0\rho^{(n)}\,\mathrm{d}x + c_0\,\mathrm{harm}_{\Omega_1^0}\left(a\rho^{(n,k)}\right)[\![\varphi]\!][\![\xi^{(n,k)}]\!].$$

Hence, we obtain the estimate

$$\begin{aligned}\liminf_{\varepsilon\downarrow 0}&\int_0^T \mathcal{R}_\varepsilon^*(\mu_\varepsilon^{(n,k)}, \xi_\varepsilon^{(n,k)})\,\mathrm{d}t\\ &\geq \int_0^T \left\{\int_{\Omega_1\setminus\Omega_1^0} \frac{1}{2}a|\partial_x\xi^{(n,k)}|^2\; c_0\rho^{(n)}\,\mathrm{d}x + \frac{1}{2}c_0\,\mathrm{harm}_{\Omega_1^0}\left(a\rho^{(n,k)}\right)[\![\xi^{(n,k)}]\!]^2\right\}\mathrm{d}t.\end{aligned}$$

Step 3. Passing to the ε-limit in the slope term: Using Jensen's estimate with respect to the measure $u^{2-m}\,\mathrm{d}t$ we observe

$$\tau_j\int_{\Omega_1^\pm} |\partial_x u^\gamma|^2\frac{1}{u^\gamma}\,\mathrm{d}x \geq \int_{\Omega_1^\pm} |\partial_x(\tau_j u)^\gamma|^2\frac{1}{(\tau_j u)^\gamma}\,\mathrm{d}x.$$

Since for $t\in(t_j,t_{j+1})$ we have that $u^n(t)$ is a convex combination of $u^n(t_j)$ and $u^n(t_{j+1})$ and the $u\mapsto|\partial_x u|^2u^{\gamma-2}$ is convex we obtain

$$\frac{|\partial_x\rho^n(t)|^2}{\rho^n(t)} \leq \frac{(t_{j+1}-t)}{(t_{j+1}-t_j)}\frac{|\partial_x\rho^n(t_{j+1})|^2}{\rho^n(t_{j+1})} + \frac{(t-t_j)}{(t_{j+1}-t_j)}\frac{|\partial_x\rho^n(t_j)|^2}{\rho^n(t_j)}.$$

Thus

$$\int_0^T\int_{\Omega_1}|\partial_x\rho^n(t)|^2\frac{1}{\rho^n(t)}\,\mathrm{d}x\,\mathrm{d}t \leq \int_0^T\int_{\Omega_1}|\partial_x\rho(t)|^2\frac{1}{\rho(t)}\,\mathrm{d}x\,\mathrm{d}t.$$

Hence, the integral on the bulk part is bounded. On the membrane we have $\partial_x R\big((\tau_j u_0^\pm)^\gamma, \chi_k([\![\tau_j\xi_0]\!])\big)\in\mathrm{L}^2\big((0,T)\times\Omega_1^0\big)$ due to the L^2 bounds $\partial_x[\![\xi_0]\!], \partial_x u^\pm\in\mathrm{L}^2(0,T)$ and the L^∞ bound $|\sinh(\chi_k([\![\tau_j\xi_0]\!]/2)|\leq\sinh(k/2)$ and $\alpha\leq u^\pm\leq\alpha^{-1}$. Thus the slope is well defined and it depends on ε in terms of the mass correction only. Hence, we obtain

$$\lim_{\varepsilon\downarrow 0}\int_0^T\mathcal{R}_\varepsilon^*\big(\mu_\varepsilon^{(n,k)}, -\mathrm{D}\mathcal{E}_\varepsilon(\mu_\varepsilon^{(n,k)})\big)\,\mathrm{d}t = \int_0^T\int_{\Omega_1}\frac{1}{2}|\partial_x\mathrm{E}_1'(\rho^{(n,k)})|^2a\rho^{(n,k)}\,\mathrm{d}\Pi_0\,\mathrm{d}t$$

Step 4. Passing to the n-limit in the continuity equation: We have strong convergence $\dot u^n\to\dot u_0$ in $\mathrm{L}^2\big(0,T;\mathrm{H}^1(\Omega_1\setminus\Omega_1^0)^*\big)$ by Lemma D.5.

Consequently, we obtain that $\xi^{(n,k)}$ is bounded in $\mathrm{L}^2\big(0,T;\mathrm{H}^1(\Omega_1\setminus\Omega_1^0)\big)$, since for φ with $\int_{\Omega_1\setminus\Omega_1^0}\varphi\,\mathrm{d}x=0$ we have that

$$\|\!|\varphi|\!\|^2 := \int_{\Omega_1\setminus\Omega_1^0} a|\partial_x\varphi|^2\; c_0\rho^{(n)}\,\mathrm{d}x + c_0\,\mathrm{harm}_{\Omega_1^0}\big(a\rho^{(n,k)}\big)\,[\![\varphi]\!]^2$$

is equivalent to the $\mathrm{H}^1(\Omega_1\setminus\Omega_1^0)$-norm. Hence, there exists a weak limit $\xi^{(k)}\in\mathrm{L}^2\big(0,T;\mathrm{H}^1(\Omega_1\setminus\Omega_1^0)\big)$. By Lebesgue dominated convergence theorem we obtain $\rho^{(n,k)}_{|(0,T)\times\Omega_1^0}\to\big(R\big(\rho_0^{\pm},\chi_k([\![\xi_0]\!])\big)\big)=:\rho^k$ in $\mathrm{L}^p(0,T\times\Omega_1^0)$ for all $p\geq 1$. In particular, $\mathrm{harm}_{\Omega_1^0}\big(a\rho^{(n,k)}\big)\to\mathrm{harm}_{\Omega_1^0}\big(a\rho^{(k)}\big)$ in $\mathrm{L}^p(0,T)$ for all $p\geq 1$. Hence, we obtain

$$\langle\varphi,\dot\mu_0\rangle = \int_{\Omega_1\setminus\Omega_1^0} a\partial_x\varphi\partial_x\xi^{(k)}\; c_0\rho_0\,\mathrm{d}x + c_0\,\mathrm{harm}_{\Omega_1^0}\big(a\rho^{(k)}\big)\,[\![\varphi]\!][\![\xi^{(k)}]\!]$$

and

$$\begin{aligned}\liminf_{n\to\infty}\int_0^T\left\{\int_{\Omega_1\setminus\Omega_1^0}\frac12 a|\partial_x\xi^{(n,k)}|^2\;\rho^{(n)}\,\mathrm{d}x+\frac12\,\mathrm{harm}_{\Omega_1^0}\big(a\rho^{(n,k)}\big)\,[\![\xi^{(n,k)}]\!]^2\right\}\mathrm{d}t\\ \geq\int_0^T\left\{\int_{\Omega_1\setminus\Omega_1^0}\frac12 a|\partial_x\xi^{(k)}|^2\;\rho_0\,\mathrm{d}x+\frac12\,\mathrm{harm}_{\Omega_1^0}\big(a\rho^{(k)}\big)\,[\![\xi^{(k)}]\!]^2\right\}\mathrm{d}t.\end{aligned}$$

Step 5. Passing to the n-limit in the slope: By strong convergence we immediately obtain

$$\begin{aligned}\lim_{n\to\infty}\int_0^T\left\{\int_{\Omega_1\setminus\Omega_1^0}\frac12 a\frac{|\partial_x\rho^{(n)}|^2}{\rho^{(n)}}\,\mathrm{d}x+\int_{\Omega_1^0}\frac12 a|\partial_x\mathrm{E}_1'(\rho^{(n,k)})|^2\rho^{(n,k)}\,\mathrm{d}x\right\}\mathrm{d}t\\ =\int_0^T\left\{\int_{\Omega_1\setminus\Omega_1^0}\frac12 a\frac{|\partial_x\rho_0|^2}{\rho_0}\,\mathrm{d}x+\int_{\Omega_1^0}\frac12|\partial_x\mathrm{E}_1'(\rho^{(k)})|^2 a\rho^{(k)}\,\mathrm{d}x\right\}\mathrm{d}t.\end{aligned}$$

Step 6. Passing to the k-limit in the continuity equation: Note that $\rho^{(k)}\nearrow R\big(\rho_0^{\pm},[\![\xi_0]\!]\big)=:\rho^0$ monotonously. Moreover, since $\dot u_0\in\mathrm{L}^2\big(0,T;\mathrm{H}^1(\Omega_1\setminus\Omega_1^0)^*\big)$ and $u_0\geq\alpha$ we have

$$\|\dot u_0\|^2_{\mathrm{L}^2\left(0,T;\mathrm{H}^1(\Omega_1\setminus\Omega_1^0)^*\right)}\geq c\int_0^T\int_{\Omega_1\setminus\Omega_1^0}|\partial_x\xi^{(k)}|^2\,\mathrm{d}x+[\![\xi^k]\!]^2\,\mathrm{d}t.$$

Hence, we have weak convergence $\xi^{(k)}\rightharpoonup\xi$ in $\mathrm{L}^2\big(0,T;\mathrm{H}^1(\Omega_1\setminus\Omega_1^0)\big)$. In particular, the duality product on the time-space cylinder converges, i.e., $\lim\limits_{k\to\infty}\langle\langle\xi^k,\dot u\rangle\rangle=\langle\langle\xi,\dot u\rangle\rangle$ and

$$\begin{aligned}\liminf_{k\to\infty}\int_0^T\mathrm{harm}_{\Omega_1^0}\big(a\rho^k\big)\,[\![\xi^k]\!]^2\;\mathrm{d}t&\geq\lim_{k_2\to\infty}\liminf_{k_1\to\infty}\int_0^T\mathrm{harm}_{\Omega_1^0}\big(a\rho^{k_2}\big)\,[\![\xi^{k_1}]\!]^2\,\mathrm{d}t\\&\geq\int_0^T\mathrm{harm}_{\Omega_1^0}\big(a\rho^0\big)\,[\![\xi]\!]^2\,\mathrm{d}t\end{aligned}$$

as well as

$$\liminf_{k\to\infty} \int_0^T \int_{\Omega_1\setminus\Omega_1^0} a|\partial_x \xi^{(k)}|^2 \ \rho_0 \,\mathrm{d}x\,\mathrm{d}t \geq \int_0^T \int_{\Omega_1\setminus\Omega_1^0} a|\partial_x \xi|^2 \rho_0 \,\mathrm{d}x\,\mathrm{d}t \geq 0.$$

Moreover, $\xi = \xi_0 \in \partial\mathcal{R}_{\text{eff}}(\mu_0, \dot\mu_0)$. This follows from [AGS05] (see Section 2.2) since $\operatorname{harm}_{\Omega_1^0}\left(a\rho^k\right)[\![\xi^k]\!]$ is equi-integrable. Passing to the limit in the continuity equation we obtain

$$\int_0^T \langle \varphi, \dot\mu_0\rangle \,\mathrm{d}t = \int_0^T \int_{\Omega_1\setminus\Omega_1^0} a\partial_x\varphi\partial_x\xi c_0\rho_0 \,\mathrm{d}x + c_0 \operatorname{harm}_{\Omega_1^0}\left(a\rho^0\right)[\![\varphi]\!][\![\xi]\!]\,\mathrm{d}t.$$

By Lemma B.3 we know that

$$\operatorname{harm}_{\Omega_1^0}\left(a\rho^0\right)[\![\xi_0]\!] = \mathsf{a}(a,\zeta)\mathscr{C}^{*\prime}([\![\xi_0]\!])\sqrt{\rho^+\rho^-}.$$

By uniqueness we conclude $\xi = \xi_0$.

Step 7. Passing to the k-limit in the slope: It remains to show that

$$\lim_{k\to\infty} \int_0^T \int_{\Omega_1^0} \frac{1}{2} a|\partial_x \mathrm{E}_1'(\rho^{(k)})|^2 \rho^{(k)} \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_{\Omega_1^0} \frac{1}{2} a|\partial_x \mathrm{E}_1'(\rho^0)|^2 \rho^0 \,\mathrm{d}x\,\mathrm{d}t.$$

Note that Lemma B.3 gives

$$\int_{\Omega_1^0} \frac{1}{2}|\partial_x \mathrm{E}_1'(\rho^{(k)}) - \partial_x\zeta|^2 a\rho^{(k)}\,\mathrm{d}z$$
$$= \mathsf{a}(a,\zeta)\Big(\mathscr{C}^*([\![\log\rho]\!]) - \mathscr{C}^*(\chi_k[\![\xi_0]\!]) + \mathscr{C}^{*\prime}(\chi_k[\![\xi_0]\!])\chi_k[\![\xi_0]\!]\Big)\sqrt{\rho^+\rho^-}.$$

Since both, $\mathscr{C}^*(\chi_k[\![\xi_0]\!])$ and $\mathscr{C}^{*\prime}(\chi_k[\![\xi_0]\!])\chi_k[\![\xi_0]\!]$ are integrable and converge monotonously we conclude step 7.

Step 8. Conclusion: Since ρ^0 is the minimizer of 5.35 we have

$$\int_{\Omega_1^0} \frac{c_0}{2}|\partial_x \mathrm{E}_1'(\rho^0) - \partial_x\zeta|^2 a\rho^0\,\mathrm{d}x - \frac{c_0}{2}\operatorname{harm}_{\Omega_1^0}\left(a\rho^0\right)[\![\xi_0]\!]^2$$
$$= \mathcal{R}^\zeta_{\text{memb}}(u_0^\pm, -\mathrm{D}\mathcal{E}_0(\mu_0)) - \mathcal{R}^\zeta_{\text{memb}}(u_0^\pm, \xi_0).$$

From step 1-7 we conclude

$$\limsup_{k\to\infty}\limsup_{n\to\infty}\limsup_{\varepsilon\downarrow 0}\mathfrak{D}^\zeta_\varepsilon(\mu_\varepsilon^{(n,k)}, [0,T]) \leq$$
$$\int_0^T \Big\{\langle\xi_0,\dot\mu_0\rangle - \mathcal{R}^*_{\text{bulk}}(\mu_0,\xi_0) - \mathcal{R}^{\zeta*}_{\text{memb}}(u_0^\pm,\xi_0) + \mathcal{R}^*_{\text{bulk}}\big(\mu_0, -\mathrm{D}\mathcal{E}_0(\mu_0)+\zeta\big)$$
$$+ \mathcal{R}^{\zeta*}_{\text{memb}}\big(u_0^\pm, -\mathrm{D}\mathcal{E}_0(\mu_0)+\zeta\big)\Big\}\,\mathrm{d}t = \mathfrak{D}^\zeta_{\text{eff}}(\mu_0,[0,T]).$$

With Lemma 1.2 the claim follows. $\square$

5.2.4 Convergence of the gradient flows

For the solutions μ_ε of the gradient flow induced by the gradient system $(\mathrm{X}, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ we find a limit μ that is the solution to the limiting EDB

$$\mathcal{E}_0\big(\mu(T)\big) + \mathfrak{D}_{\mathrm{eff}}(\mu_0, [0,T]) = \mathcal{E}_0\big(\mu(0)\big).$$

This can be deduced from the same methods as in Section 4.2. However, since the Boltzmann entropy allows us to conclude pointwise in time BV-regularity it is not obvious how to show

$$\hat{\mu}(0) = \lim_{\varepsilon \downarrow 0} \mu_\varepsilon(0) \overset{!}{=} \lim_{t \downarrow 0} \mu(t) = \mu(0)$$

using the information of $\mathcal{E}_\varepsilon$ and $\mathcal{R}_\varepsilon$ only. But we have almost everywhere convergence of the energies. In particular, we find for ζ such that $\mathsf{a}(a,\zeta) = \mathrm{harm}_{\Omega_1^0}(a)$ that we have almost EDP convergence with tilting except the fact that we can only conclude almost everywhere convergence of the solutions to the limit solution and energies.

The limiting equation reads

$$\dot{u} = \mathrm{div}(a\partial_x u^\gamma) \quad \text{on } \Omega_1 \setminus \Omega_1^0$$

with the transmission condition

$$\big(a\partial_x u^\gamma\big)(x_\pm) = \mathrm{harm}_{\Omega_1^0}(a)\, [\![u^\gamma]\!]$$

and the boundary conditions

$$\big(a\partial_x u^\gamma\big)(\pm 1 + x_\pm) = 0.$$

Note that this is exactly the limiting equation of Section 5.1 with $\gamma = m$ and $\mathsf{m}(u) = u^1$.

6 Summary

In chapters 3-5 we investigate the Γ-convergence of the tilted total dissipation functional to either conclude relaxed EDP-convergence or EDP-convergence with tilting. We obtained a relaxed EDP-convergence result for the gradient system considered in Chapter 3. Thus the motivation of EDP-convergence with tilting (see Definition 1.10) is justified. Moreover, the dependence of the effective dissipation potential on the wiggly part of the energy is given explicitly. The problems of chapters 4 and 5 featured thin layers instead of wiggly energies. In Section 4.1 we had a doubly quadratic gradient structure in the sense that the energy and the dissipation potential are quadratic. As a result this property is preserved under Γ-convergence in contrast to sections 4.2-5.2. Here we have non-quadratic energies and state dependent dissipation potentials.

The limiting dissipation potentials of sections 4.2-5.2 depend on the tilt restricted to the vanishing layer. Thus we consider the limiting dissipation potentials as independent of the effective force associated with the tilt which is the jump of the tilt across the vanishing layer. However, for sections 4.2 and 5.2 we can identify a class of tilts such that the limiting dissipation potentials do not depend on the tilt.

As a byproduct of the limiting gradient structures we obtained the limiting equations and the convergence of solutions to the solution of the limit gradient system.

Variant of Poincaré's estimate

Let $d > 1$ and $\Omega_1 = \Sigma \times I \subset \mathbb{R}^d$ be a domain with a domain $\Sigma \subset \mathbb{R}^{d-1}$ and and interval $I \subset \mathbb{R}$. We donote $\nabla_y = (\partial_1, \dots, \partial_{d-1})$ and $\partial_z = \partial_d$. Let I^+, I^0, I^- be subintervals with $I = I^+ \cup \bar{I}^0 \cup I^-$. The subdomains Ω_1^ι are defined via $\Omega_1^\iota = \Sigma \times I^\iota$ for $\iota \in \{+, 0, -\}$. Moreover, let $\mathbf{m}_\varepsilon = 1$ on $\Omega_1 \setminus \Omega_1^0$ and $\mathbf{m}_\varepsilon = \varepsilon$ on Ω_1^0. We have the following Poincaré-Wirtinger estimate.

Lemma A.1. *There exists $c > 0$ such that for all $\varepsilon > 0$ and for all $\xi \in \mathrm{H}^1(\Omega_1)$ satisfying $\int_{\Omega_1} \xi \mathbf{m}_\varepsilon \,\mathrm{d}x = 0$ it holds*

$$\int_{\Omega_1} \xi^2 \mathbf{m}_\varepsilon \,\mathrm{d}x \leq c \int_{\Omega_1} |\nabla_y \xi|^2 \mathbf{m}_\varepsilon + |\partial_d \xi|^2 \,\mathrm{d}x.$$

Proof. We assume the opposite. Let ξ_ε such that $\int_{\Omega_1} \xi_\varepsilon^2 \mathbf{m}_\varepsilon \,\mathrm{d}x = 1$ and

$$\varepsilon \geq \int_{\Omega_1} |\nabla' \xi_\varepsilon|^2 \mathbf{m}_\varepsilon + |\partial_d \xi_\varepsilon|^2 \,\mathrm{d}x.$$

In particular, $\xi_{\varepsilon|_{\Omega_1^\pm}} \to c_\pm$ in $\mathrm{H}^1(\Omega_1^\pm)$ with some $c_\pm \in \mathbb{R}$. Since additionally $(\partial_d \xi_\varepsilon)_{|_{\Omega_1^0}} \to 0$ in $\mathrm{L}^2(\Omega_1^0)$ we conclude that $c_+ = c_-$. Due to the constraint $\int_{\Omega_1} \xi \mathbf{m}_\varepsilon \,\mathrm{d}x = 0$ we obtain $c_\pm = 0$. This is in contradiction to $\int_{\Omega_1} \xi^2 \mathbf{m}_\varepsilon \,\mathrm{d}x = 1$. $\square$

By an identical proof for $d = 1$, i.e., $\Omega_1 = I$ and $\nabla_y = 0$, we obtain the following lemma.

Lemma A.2. *There exists $c > 0$ such that for all $\varepsilon > 0$ and for all $\xi \in \mathrm{H}^1(\Omega_1)$ satisfying $\int_{\Omega_1} \xi \mathbf{m}_\varepsilon \,\mathrm{d}x = 0$ it holds*

$$\int_{\Omega_1} \xi^2 \mathbf{m}_\varepsilon \,\mathrm{d}x \leq c \int_{\Omega_1} |\partial_d \xi|^2 \,\mathrm{d}x.$$

Explicit solution

In Section 4.2 and Section 5.2 the minimization problem

$$\mathcal{F}(r) = \inf_u \left\{ \int_{I_1^0} a|\partial_x \log u - \partial_x \zeta|^2 u \, \mathrm{d}z - \operatorname{harm}_{I_1^0}(au)\, r^2 \right\} \tag{B.1}$$

subject to the boundary conditions $u(z^\pm) = u^\pm$ occurred. Due to (5.34) we know already the minimal value (cf. [LMPR17, Prop A.3]), i.e.

$$\mathcal{F}(r) = 2\operatorname{harm}\left(a\mathrm{e}^\zeta\right) \sqrt{u^+u^-\mathrm{e}^{-(\zeta^+ + \zeta^-)}}\left(\mathscr{C}^*(\llbracket \log(u) - \zeta \rrbracket) - \mathscr{C}^*(r)\right).$$

We introduce the change of variables via

$$X(z) = \operatorname{harm}_{I_1^0}\left(a\mathrm{e}^\zeta\right) \int_{z^-}^{z} \frac{1}{a\mathrm{e}^\zeta} \, \mathrm{d}\hat{z} + z^-.$$

Since $(z^+ - z^-) = 1$ we have $X(z^\pm) = z^\pm$. Moreover, X is invertible. With $X^{-1} =: Z$ we compute $\mathrm{d}z = Z'(x)\,\mathrm{d}x$ and $\partial_z = 1/Z'(x)\partial_x$ as follows:

$$Z'(x) = \frac{1}{X'(Z(x))} \quad \text{and} \quad X'(z) = \operatorname{harm}_{I_1^0}\left(a\mathrm{e}^\zeta\right) \frac{1}{a(z)\mathrm{e}^\zeta(z)}.$$

Thus we arrive at

$$\mathcal{F}(r) = \operatorname{harm}_{I_1^0}\left(a\mathrm{e}^\zeta\right) \inf_w \left\{ \int_{I_1^0} |\partial_x \log w|^2 w \, \mathrm{d}x - \operatorname{harm}_{I_1^0}(w)\, r^2 \right\}$$

with $w = (u\mathrm{e}^{-\zeta}) \circ Z$. The solution is given explicitly in the following lemma.

Lemma B.3.

$$w_0(x) = \left(w^+ + w^- - 2\sqrt{w^+w^-}\cosh(r/2)\right)(x^2 - 1/4) + \llbracket w \rrbracket x + (w^+ + w^-)/2$$

satisfies the boundary conditions and

$$2\sqrt{w^+w^-}\left(\mathscr{C}^*(\llbracket \log(w) \rrbracket) - \mathscr{C}^*(r)\right) = \int_{I_1^0} |\partial_x \log w_0|^2 w_0 \, \mathrm{d}x - \operatorname{harm}_{I_1^0}(w_0)\, r^2.$$

Moreover,

$$\operatorname{harm}_{I_1^0}(w_0) = 2\sqrt{w^+w^-}\sinh(r/2)/r.$$

Proof. Since

$$2\mathrm{Artanh}(z) = \ln\left(\frac{1+z}{1-z}\right) \quad \text{and} \quad \mathrm{Artanh}'(z) = \frac{1}{1-z^2}$$

we compute that

$$\partial_x 2\mathrm{Artanh}\left(\frac{-b-2ax}{\sqrt{a^2+b^2-4ac}}\right)\frac{1}{\sqrt{a^2+b^2-4ac}} = \frac{1}{a(x^2-1/4)+bx+c}.$$

With $a = w^+ + w^- - 2\sqrt{w^+w^-}\cosh(r/2)$, $b = [\![w]\!]$ and $c = (w^+ + w^-)/2$ we calculate

$$\sqrt{a^2+b^2-4ac} = 2\sqrt{w^+w^-}\sinh(|r|/2)$$

and

$$2\mathrm{Artanh}\left(\frac{-b-a}{\sqrt{a^2+b^2-4ac}}\right) - 2\mathrm{Artanh}\left(\frac{-b+a}{\sqrt{a^2+b^2-4ac}}\right) = |r|.$$

Thus $\mathrm{harm}_{I_1^0}(w_0)\, r^2 = 2\sqrt{w^+w^-}\, r\sinh(r/2)$ and

$$\int_{I_1^0} |\partial_x \log w_0|^2 w_0 \,\mathrm{d}x = 4a + \int_{I_1^0} \frac{a^2+b^2-4ac}{w_0}\,\mathrm{d}x = 4a + 2\sqrt{w^+w^-}\, r\sin(r/2).$$

We conclude via the identity

$$4a = 2\sqrt{w^+w^-}\big(\mathscr{C}^*([\![\log(w)]\!]) - \mathscr{C}^*(r)\big).$$

□

As a consequence, we find that

$$u(z) = w_0\big(X(z)\big)\mathrm{e}^{-\zeta}(z) \tag{B.2}$$

is the solution to the original minimization problem (B.1).

Existence of optimal profile

For an interval $\Omega \subset \mathbb{R}$ we used in Lemma 5.10 the existence of solutions to

$$\partial_x\big((\partial_x \mathrm{E}_m'(u) - \varphi)\mathsf{m}(u)\big) = 0 \tag{C.3}$$

with Dirichlet boundary conditions $u_{|\partial\Omega} = u_{Dir}$ and $\varphi \in \mathrm{L}^\infty(\Omega)$, where with $\mathrm{E}_m'' = u^{m-2}$ and $\mathsf{m}(u) = u^\gamma$. Indeed, using the theory of pseudo-monotone operators we show that solutions to (C.3) exist. We rewrite the problem in terms of $v + v_a := u^{m-1+\gamma}$ and obtain for $\Omega \subset \mathbb{R}$

$$\partial_x\big((\partial_x v - \varphi)(\max\{0, v + v_a\})^\beta\big) = 0 \tag{C.4}$$

with $0 < \beta = \gamma/(m-1+\gamma) < 1$, $v \in \mathrm{H}_0^1$ and $v_a \in \mathrm{H}_{Dir}^1$ is an affine function satisfying the corresponding boundary conditions. Note that we extend $v \mapsto v^\beta$ for negative values via $\max 0, v^\beta$.

Lemma C.4. *Let* $m > 1$, $0 < \gamma \leq 1$ *and* $2 \leq m + \gamma$. *Then there exists a solution* $v_0 \in$ *to* (C.4).

Proof. First we show that $T : \mathrm{H}_0^1 \ni v \mapsto \partial_x\big((\partial_x v - \varphi)(\max\{0, v + v_a\})^\beta\big) \in \mathrm{H}_0^1(\Omega)^*$ is pseudo-monotone, i.e.,

$$\text{if } v_n \rightharpoonup v \quad \text{and} \quad \limsup \langle T(v_n), v_n - v\rangle \leq 0$$

then we have for all $w \in \mathrm{H}_0^1$ that

$$\liminf \langle T(v_n), v_n - v\rangle \geq \langle T(v), v - w\rangle.$$

Due to the compact embedding $\mathrm{H}^1(\Omega) \subset \mathrm{L}^2(\Omega)$ and $\beta < 1$ we obtain that

$$\limsup \langle T(v_n), v_n - v\rangle \leq 0$$

implies strong convergence of v_n and thus we obtain $\liminf \langle T(v_n), v_n - v\rangle \geq \langle T(v), v - w\rangle$ for all $w \in \mathrm{H}_0^1$.

The next step is to show coercivity of T. We compute

$$\langle T(v), v\rangle = \|\partial_x v\|_{\mathrm{L}^2(\Omega)}^2 - \int_\Omega \varphi v^\beta \partial_x v \,\mathrm{d}x \geq \|\partial_x v\|\big(\|\partial_x v\| - \|\varphi\|_\infty(\|v\|_\infty^\beta + \max\{v_+, v_-\}^\beta)\big).$$

Since $0 < \beta < 1$ we obtain $(\|\partial_x v\| - \|\varphi\|_\infty \|v\|_\infty^\beta) \to \infty$ as $\|\partial_x v\| \to \infty$, i.e., T is coercive. Hence, there exists a solution v satisfying (C.4) (see [Rou05, Thm 2.6]). □

In order to prove existence of solutions such that $v \geq 0$ we easily verify that for any solution v we have that $\max\{0, v\}$ is also a solution.

Convergence of time discretization

In sections 4.2,5.1 and 5.2 we used the following time discretization. Let $t_j = \frac{j}{n}T$ for $0 \le j \le n$ and $t_j = 0$ for $j < 0$ and $t_j = T$ for $j > n$. The mean τ_j for $j \in \{0, \ldots, n\}$is defined via

$$\tau_j h := \fint_{t_{j-1/2}}^{t_{j+1/2}} h \,\mathrm{d}t.$$

For u the piecewise affine approximation u^n is defined at the nodal points $\{t_j\}$ via

$$u^n(t_j) = \tau_j u.$$

We prove the convergence in an abstract setting. Let X be a separable Banach space and all integrals are defined as Bochner integrals.

We show that $\dot{u}^n \to \dot{u}$ in $\mathrm{L}^2(0,T;X)$. We calculate for $j \in \{1, \ldots, n-2\}$ and $t \in (t_j, t_{j+1})$

$$\begin{aligned}
\dot{u}^n &= \left(\frac{n}{T}\right)^2 \left(\int_{t_{j+1/2}}^{t_{j+3/2}} u \,\mathrm{d}s - \int_{t_{j-1/2}}^{t_{j+1/2}} u \,\mathrm{d}s \right) \\
&= \left(\frac{n}{T}\right)^2 \int_{t_j}^{t_{j+1}} \Big\{ u\big(s + T/(2n)\big) - u\big(s - T/(2n)\big) \Big\} \,\mathrm{d}s = \fint_{t_j}^{t_{j+1}} \fint_{s-T/(2n)}^{s+T/(2n)} \dot{u} \,\mathrm{d}s.
\end{aligned}$$

Similarly, we compute

$$\dot{u}^n_{|(0,T/n)} = \fint_0^{t_1} \frac{n}{T} \int_{s/2}^{s+T/(2n)} \dot{u} \,\mathrm{d}s \quad \text{and} \quad \dot{u}^n_{|(T(n-1)/n,T)} = \fint_{t_{n-1}}^{T} \frac{n}{T} \int_{s-T/(2n)}^{(s+T)/2} \dot{u} \,\mathrm{d}s.$$

For $s \in (t_1, t_{n-1})$ we define

$$\tau_s h := \frac{n}{T} \int\limits_{s-T/(2n)}^{s+T/(2n)} h \,\mathrm{d}\sigma$$

and for $s \in (t_0, t_1)$ resp. $s \in (t_{n-1}, t_n)$ we define

$$\tau_s h := \frac{n}{T} \int\limits_{s/2}^{s+T/(2n)} h \,\mathrm{d}\sigma \quad \text{resp.} \quad \tau_s h := \frac{n}{T} \int\limits_{s-T/(2n)}^{(s+T)/2} h \,\mathrm{d}s.$$

Hence, we see that $\dot{u}^n_{|(t_j,t_{j+1})} = \tau_j\{s \mapsto \tau_s \dot{u}\}$. We denote $h = T/n$ and Π_h the projection from L^2 to the piecewise constant functions, i.e., $(\Pi_h \dot{u})_{|(t_j,t_{j+1})} = \tau_j \dot{u}$. In particular, $\Pi_h \circ \Pi_h = \Pi_h$.

Lemma D.5. *It holds*
$$\int_0^T \|\dot u^n - \dot u\|^2 \,\mathrm{d}t \to 0.$$

Proof. We estimate

$$\begin{aligned}
\int_0^T \|\dot u^n - \dot u\|^2 \,\mathrm{d}t &\le 2\int_0^T \|\dot u^n - \Pi_h \circ \Pi_h \dot u\|^2 + \|\Pi_h \dot u - \dot u\|^2 \,\mathrm{d}t \\
&\le 2\int_0^T \|\tau_t \dot u - \Pi_h \dot u\|^2 + \|\Pi_h \dot u - \dot u\|^2 \,\mathrm{d}t \\
&\le \int_{t_0}^{t_1} \frac{t+h}{h^2} \int_{t/2}^{t+h/2} \|\dot u(s) - h\tau_j \dot u\|^2 \,\mathrm{d}s \,\mathrm{d}t \\
&\quad + \int_{t_{n-1}}^{t_n} \frac{T-t+h}{h^2} \int_{t-h/2}^{(t+T)/2} \|\dot u(s) - h\tau_j \dot u\|^2 \,\mathrm{d}s \,\mathrm{d}t \\
&\quad + 2\sum_{j=1}^{n-2} \int_{t_j}^{t_{j+1}} \frac{1}{h} \int\limits_{t-h/2}^{t+h/2} \|\dot u(s) - \tau_j \dot u\|^2 \,\mathrm{d}s + \|\tau_j \dot u - \dot u\|^2 \,\mathrm{d}t.
\end{aligned}$$

Clearly, the boundary terms vanish in the limit as $h \downarrow 0$. We continue to estimate

$$\begin{aligned}
&\sum_{j=1}^{n-2} \int_{t_j}^{t_{j+1}} \frac{1}{h} \int_{t-h/2}^{t+h/2} \|\dot u(s) - \tau_j \dot u\|^2 \,\mathrm{d}s \,\mathrm{d}t \le \sum_{j=1}^{n-2} \int_{t_{j-1/2}}^{t_{j+3/2}} \|\dot u(s) - \tau_j \dot u\|^2 \,\mathrm{d}s \\
&= \sum_{j=1}^{n-2} \int_{t_{j-1/2}}^{t_j} \|\dot u(s) - \tau_j \dot u\|^2 \,\mathrm{d}s + \int_{t_{j+1}}^{t_{j+3/2}} \|\dot u(s) - \tau_j \dot u\|^2 \,\mathrm{d}s + \int_{t_j}^{t_{j+1}} \|\tau_j \dot u - \dot u\|^2 \,\mathrm{d}t \\
&\le \sum_{j=1}^{n-2} \int_{t_{j-1/2}}^{t_j} 2\|\dot u(s) - \dot u(s+h/2)\|^2 \,\mathrm{d}s + \int_{t_{j+1}}^{t_{j+3/2}} 2\|\dot u(s) - \dot u(s-h/2)\|^2 \,\mathrm{d}s \\
&\quad + \int_{t_j}^{t_{j+1}} 3\|\tau_j \dot u - \dot u\|^2 \,\mathrm{d}t \\
&\le \int_0^{T-h} \|\dot u - \mathbf{T}_{h/2} \dot u\|^2 \,\mathrm{d}t + \int_h^T \|\dot u - \mathbf{T}_{-h/2} \dot u\|^2 \,\mathrm{d}t + \int_0^T 3\|\dot u - \Pi_h \dot u\|^2 \,\mathrm{d}t
\end{aligned}$$

where $\mathbf{T}_{\pm h/2} : \{s \mapsto u(s)\} \mapsto \{s \mapsto u(s \pm h/2)\}$ is the translation operator. Since both, the translation operator and the projection are strongly continuous in h, i.e. for all $\dot u \in \mathrm{L}^2(0, T; X)$ it holds $\lim_{h\downarrow 0} \mathbf{T}_h \dot u = \lim_{h\downarrow 0} \Pi_h \dot u = \dot u$. This follows from the Lebesgue differentiation theorem (see [Hyt16, Cor 2.3.5]) and the fact that $\|\Pi_h \dot u\| \le \|\dot u\|$. □

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Selbstständigkeitserklärung

Ich erkläre, dass ich die Dissertation selbständig und nur unter Verwendung der von mir gemäß § 7 Abs. 3 der Promotionsordnung der Mathematisch-Naturwissenschaftlichen Fakultät, veröffentlicht im Amtlichen Mitteilungsblatt der Humboldt-Universität zu Berlin Nr. 126/2014 am 18. November 2014 angegebenen Hilfsmittel angefertigt habe.

Berlin, 19. Februar 2019 ....

Thomas Frenzel